1. Introduction

In this paper, we adopt the techniques in [2] to study the behavior of the continuous time *latent voter model*, which was considered by [5] in the mean field setup, on a random r-regular graph on n vertices. We choose the random graph G_n on the vertex set $[n] := \{1, 2, \dots, n\}$ according to the uniform distribution $\tilde{\mathbb{P}}$ on simple graphs, and once chosen the graph remains fixed through time.

We write $x \sim y$ to mean that x is a neighbor of y, and let

$$\mathcal{N}_y := \{x \in [n] : x \sim y\} \tag{1.1}$$

N

be the set of neighbors of y. The distribution $P_{G_n,\lambda}$ of the latent voter model with parameter λ conditioned on G_n can be described as follows. At any time each vertex is either active or inactive, and $\xi_t(x) \in \{0, 1\}$ denotes one of the two possible opinions of x at time t. Initially all the vertices are active. When a vertex is active, at rate 1 it adopts the opinion of an uniformly chosen random neighbor. Change in opinion makes an active vertex inactive. In the inactive phase, a vertex does not change its opinion and returns to active phase at rate λ. Let A_t be the set of active vertices and $\boldsymbol{\xi}_t := \{v : \xi_t(v) = 0\}$ be the set of vertices with opinion 0 at time t. If $\mathbf{P}_\lambda$ denotes the distribution of the latent voter model on the random graph G_n having distribution $\tilde{\mathbb{P}}$, then

$$\mathbf{P}_\lambda(\cdot) = \tilde{\mathbb{E}} P_{G_n,\lambda}(\cdot),$$

where $\tilde{\mathbb{E}}$ is the expectation corresponding to the probability distribution $\tilde{\mathbb{P}}$. In this paper we consider $\lambda = \lambda_n$ such that $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$. Here and later $a_n \ll b_n$ (or equivalently $b_n \gg a_n$) means $a_n/b_n \to 0$ as $n \to \infty$.

2. Construction and duality

2.1. **Construction of the graph G_n.** We construct our random graph G_n on the vertex set $[n] := \{1, 2, \dots n\}$ by assigning r "half-edges" to each of the vertices, and then pairing the half-edges at random. If r is odd, then n must be even so that the number of half-edges, rn, is even to have a valid degree sequence. Let $\mathbb{P}$ denote this distribution of G_n. We condition on the event E_n that the graph is simple, i.e. it does not contain a self-loop at any vertex, or more than one edge between two vertices. It can be shown (see e.g. Corollary 9.7 on page 239 of [?JLR00]) that $\mathbb{P}(E_n)$ converges to a positive limit as $n \to \infty$, and hence

$$\text{if } \tilde{\mathbb{P}} := \mathbb{P}(\cdot | E_n), \text{ then } \tilde{\mathbb{P}}(\cdot) \le c\mathbb{P}(\cdot) \text{ for some constant } c = c(r) > 0. \tag{2.1}$$

Ptilde

So the conditioning on the event E_n will not have much effect on the distribution of G_n. It is easy to see that the distribution of G_n under $\tilde{\mathbb{P}}$ is uniform over the collection of all undirected r-regular graphs on the vertex set $[n]$. Let

$$\begin{aligned} &d(u,v) \text{ be the length of the shortest path between } u \text{ and } v, D(v,M) = \{u : d(u,v) \le M\} \\ &d(U,v) := \min_{u \in U} d(u,v) \text{ and } G_{v,M} \text{ be the subgraph of } G_n \text{ induced by } D(v,M). \end{aligned} \tag{2.2}$$

We call v and M to be the root and depth of $G_{v,M}$ respectively. If $G_{x,M}$ has no loop, then it is a finite r-tree, *i.e.,* all the vertices except the leaves have degree r.

If we let

$$L_i(G_n) := \left\{ v \in [n] : G_{v,\lceil (1/5)\log_{r-1} n \rceil} \text{ has at most } i \text{ loops} \right\}, i = 0, 1, \text{ and} \tag{2.3}$$

LOG

$$\begin{aligned} \mathcal{G}_n := \{&\text{simple } r\text{-regular graph } G_n \text{ on the vertex set } [n] : |L_0(G_n)| \ge n - 2n^{4/5} \\ &\text{and } |L_1(G_n)| = n\}, \end{aligned} \tag{2.4}$$

cG

then the following probability estimates show that $\tilde{\mathbb{P}}(G_n \in \mathcal{G}_n) \to 1$ as $n \to \infty$.

graph est **Lemma 2.1.** *For $L_0(G_n)$ and $\mathcal{G}_n$ as in (2.3) and (2.4),*

$(1)\tilde{\mathbb{P}}(v \notin L_0(G_n)) \le C^1_{2.1} n^{-3/5} \quad (2)\tilde{\mathbb{P}}(\mathcal{G}_n^c) \le C^2_{2.1} n^{-1/5} for some constants C^i_{2.1} = C^i_{2.1}(r) > 0.$

Proof. While exploring the vertices of G_n one at a time starting from v and using a breadth-first search algorithm based on the distance function d of (2.2), the maximum number of vertices in $G_{v,\lceil (1/5)\log_{r-1} n\rceil}$ is $r[1 + (r-1) + \cdots + (r-1)^{\lceil (1/5)\log_{r-1} n\rceil - 1}] \le 2rn^{1/5}$. So under the law $\mathbb{P}$ at any step the probability of selecting a vertex that has already been touched is $\le 2r^2 n^{1/5}/(rn - 2r^2 n^{1/5})$. So,

$$\tilde{\mathbb{P}}(v \notin L_0(G_n)) \;\le\; c(r)\mathbb{P}(v \notin L_0(G_n)) \le c(r)\frac{2r^3 n^{2/5}}{rn - 2r^2 n^{1/5}} \le c_1(r) n^{-3/5}, \tag{2.5}$$

$$\tilde{\mathbb{P}}(v \notin L_1(G_n)) \;\le\; c(r)\mathbb{P}(v \notin L_1(G_n)) \le c(r)\left(\frac{2r^3 n^{2/5}}{rn - 2r^2 n^{1/5}}\right)^2 \le c_2(r) n^{-6/5}$$

for large enough n. Hence, $\tilde{\mathbb{P}}(|L_1(G_n)| < n) \le n \cdot c_2(r) n^{-6/5} = c_2(r) n^{-1/5}$.

Next observe that if Cl_v and Cl_w are the clusters of size $2rn^{1/5}$ starting from v and w respectively and $Cl_{v,w} = Cl_v \cap Cl_w$, then using similar argument as above

$$\tilde{\mathbb{P}}(Cl_{v,w} \ne \emptyset) \le c(r)\frac{2r^3 n^{2/5}}{rn - 2r^2 n^{1/5}} \le c_1(r) n^{-3/5} \text{ for large enough } n.$$

Since on the event $\{Cl_{v,w} = \emptyset\}$, $\{v \in L_0(G_n)\}$ is independent of $\{w \in L_0(G_n)\}$,

$$\tilde{\mathbb{P}}(v, w \notin L_0(G_n)) \le \tilde{\mathbb{P}}(Cl_{v,w} \ne \emptyset) + \tilde{\mathbb{P}}(v, w \notin L_0(G_n), Cl_v \cap Cl_w = \emptyset)$$

$$\le \tilde{\mathbb{P}}(Cl_{v,w} \ne \emptyset) + \frac{\tilde{\mathbb{P}}(\{v \notin L_0(G_n)\} \cap \{Cl_v \cap Cl_w = \emptyset\})\tilde{\mathbb{P}}(\{w \notin L_0(G_n)\} \cap \{Cl_{v,w} = \emptyset\})}{\tilde{\mathbb{P}}(Cl_{v,w} = \emptyset)},$$

so that $cov_{\tilde{\mathbb{P}}}(\mathbf{1}_{\{v \notin L_0(G_n)\}}, \mathbf{1}_{\{w \notin L_0(G_n)\}}) \le 3\tilde{\mathbb{P}}(Cl_{v,w} \ne \emptyset) \le 3c_1(r) n^{-3/5}$ for large enough n.

Using the above estimate and a standard second moment argument we get (2). ∎

2.2. **Construction of ξ_t.** Since we will mostly work with the rescaled process $\xi_t^{\lambda_n}, t \ge 0$, where $\xi_t^{\lambda_n} := \xi_{\lambda_n t}$, here we describe the construction of the process $\xi_t^{\lambda_n}$. To construct the process, we use a graphical representation. For $x \in [n]$, we set $W_0^x = 0 = V_0^x$ and introduce independent Poisson processes $W^x := \{W_m^x : m \ge 1\}$ and $V^x := \{V_m^x : m \ge 1\}$ with rates λ_n^2 and λ_n respectively. From the continuity of the exponential distribution it is easy to see that

$$P_{G_n,\lambda_n}(V^x \cap W^y = \emptyset \text{ for all } x, y \in [n]) = 1.$$

For each $x \in [n]$, we put a dot at the locations $(x, W_m^x), m \ge 0$, and call them 'wake up dots' for x. At the times W_m^x, the vertex x becomes active irrespective of its earlier status. We call the times $V_m^x, m \ge 1$, *voting times* for x. If x is active at some voting time V_m^x, it consults with a random neighbor $Y_{x,m}$ having uniform distribution over $\mathcal{N}_x$ to consider whether to change opinion or not. So the value $\xi_t^{\lambda_n}(x)$ do not change for $t \in [V_{m-1}^x, V_m^x)$. The random variables $Y_{x,m}, x \in [n]$, are independent of the Poisson processes and all are independent of an initial configuration $\xi_0^{\lambda_n} \in \{0,1\}^{[n]}$.

We obtain the values $\xi_t^{\lambda_n}(x)$ recursively as follows. First we partition the voting times for x according to their positions relative to the corresponding wake up dots. For $m \ge 1$, let

$$I_m^x := V^x \cap [W_{m-1}^x, W_m^x). \tag{2.6}$$ I_x^m def

So I_m^x denotes the voting times for x between its $(m-1)^{th}$ and m^{th} wake up dots.

$$I.\ \text{If } I_m^x = \emptyset, \text{ then we set } \xi_t^{\lambda_n}(x) = \xi_{W_{m-1}^x}^{\lambda_n}(x) \text{ for } t \in [W_{m-1}^x, W_m^x] \tag{2.7}$$ xidef1

and x is active during $[W_{m-1}^x, W_m^x]$.

$$II.\ \text{If } I_m^x = \{V_k^x\}, \text{ then we set } \xi_t^{\lambda_n}(x) = \begin{cases} \xi_{W_{m-1}^x}^{\lambda_n}(x) \text{ for } t \in [W_{m-1}^x \le t < V_k^x \\ \xi_{V_k^x-}^{\lambda_n}(Y_{x,k}) \text{ for } V_k^x \le t \le W_m^x \end{cases} \tag{2.8}$$ xidef2

and x becomes inactive at time V_k^x if and only if $\xi_{W_{m-1}^x}^{\lambda_n}(x) \neq \xi_{V_k^x}^{\lambda_n}(x)$.

In case *II*, we call V_k^x a *single voting time* and W_{m-1}^x a *single wake up dot* for x. To facilitate the definition of the dual process we draw an arrow from (x, V_k^x) to $(Y_{x,k}, V_k^x)$ and call it a voter arrow. For $|I_m^x| \ge 2$ there are two cases depending on

$$J_m^x := \left\{ j : V_j^x \in I_m^x \text{ with } \xi_{W_{m-1}^x}^{\lambda_n}(x) \neq \xi_{V_j^x}^{\lambda_n}(Y_{x,j}) \right\}.$$

Let $j_m^x := \min J_m^x$ when $J_m^x \neq \emptyset$.

$$III.\ \text{If } |I_m^x| \ge 2 \text{ and } J_m^x \neq \emptyset, \text{then we set } \xi_t^{\lambda_n}(x) = \begin{cases} \xi_{W_{m-1}^x}^{\lambda_n}(x) \text{ for } W_{m-1}^x \le t < V_{j_m^x}^x \\ 1 - \xi_{W_{m-1}^x}^{\lambda_n}(x) \text{ for } V_{j_m^x}^x \le t \le W_m^x, \end{cases} \tag{2.9}$$ xidef3

and x becomes inactive at the time $V_{j_m^x}^x$.

$$IV.\ \text{If } |I_m^x| \ge 2 \text{ and } J_m^x = \emptyset, \text{ then we set } \xi_t^{\lambda_n}(x) = \xi_{W_{m-1}^x}^{\lambda_n}(x) \text{ for } t \in [W_{m-1}^x, W_m^x] \tag{2.10}$$ xidef4

and x remains active during $[W_{m-1}^x, W_m^x]$.

In other words, at the voting times $V_j^x \in I_m^x$, x adopts the opinion of $Y_{x,j}$ for $j \le j_m^x$ and ignores the opinion of $Y_{x,j}$ for $j > j_m^x$. In cases *III* and *IV*, we need to know the state of the vertices x at time W_{m-1}^x and $Y_{x,k}$ at time $V_{x,k}$ for $V_k^x \in I_m^x$ to update that of x during the time interval $[W_{m-1}^x, W_m^x]$. So in order to facilitate the definition of the dual, we write a $*$ next to (x, W_{m-1}^x), call W_{m-1}^x a $*$-*dot*, draw an arrow from (x, W_{m-1}^x) to each of $(Y_{x,k}, W_{m-1}^x)$ for $k \in \{k : V_k^x \in I_m^x\}$, and call these $*$-arrows.

It is not hard to show that the above recipe defines a pathwise unique process. To compute the state of a vertex at time T we work backwards in time and use the following approximate dual process.

At times it is easier to use notation for the independent Poisson processes of voting events $\Lambda_v^x(dt, dy), x \in [n]$, on $\mathbb{R} \times \mathcal{N}_x$ with points $\{(V_m^x, Y_{x,m})\}$ and intensity $\lambda_n\, dt \times \theta_x$, where θ_x is uniform over $\mathcal{N}_x$, and also the independent Poisson processes of wake up events $\Lambda_w^x(dt), x \in [n]$, on $\mathbb{R}$ with points $\{W_m^x\}$ and intensity $\lambda_n^2\, dt$.

sect_X

2.3. The approximate dual process X.

Fix $T > 0$ and a vector of $M+1$ vertices $\mathbf{z} = (z_0, \ldots, z_M)$, where each $z_i \in [n]$. The dual process $\mathbf{X} = \mathbf{X}^{\mathbf{z},T}$ starts from these vertices at time T and then works backwards in time to determine the values $\xi_T^{\lambda_n}(z_i)$. $\mathbf{X}$ will be a *coalescing branching random walk* taking values in

$$\mathcal{D} := [D([0,T], [n] \cup \{\infty\})]^{\mathbb{N}}$$

and starting from $\mathbf{X}_0 = (z_0, \ldots, z_M, \infty, \infty, \ldots)$. Here $D([0,T], [n] \cup \{\infty\})$ denotes the set of all cádlág paths $\omega : [0,T] \to [n] \cup \{\infty\}$ endowed with Skorokhod topology, and $\mathcal{D}$ is given the product topology.

For $\mathbf{X}^{\mathbf{z},T} = (\mathbf{X}^{\mathbf{z},T,0}, \mathbf{X}^{\mathbf{z},T,1}, \dots) \in \mathcal{D}$, let $k^{\mathbf{z}}(t) = k(t) := \max\{i : X_t^i \neq \infty\}$. Define an equivalence relation $\sim_t$ on $\{0, 1, \dots, k(t)\}$ as $i \sim_t i'$ if $X_t^{\mathbf{z},T,i} = X_t^{\mathbf{z},T,i'} \neq \infty$, and choose the minimum index from each of the equivalent classes to form the set $J^{\mathbf{z}}(t) = J(t)$. We need to know the states of the vertices in $\{X_t^i : i \in J(t)\}$ at time $T - t$ to determine the states of $z_0, \dots, z_M$ at time T. We often drop the superscripts $\mathbf{z}, T$ when there is no ambiguity.

If there were no $*$-arrows, then the coordinates $X_t^j, j \in J(t)$, follow the system of coalescing random walks. Coalescing refers to the fact that if $X_s^j = X_s^{j'} \neq \infty$ for some $s < T$ and $j, j' \leq k(s)$, then $X_t^j = X_t^{j'}$ for all $t \in [s, T]$. Jumps in the coalescing random walk system occurs when one of the particles in the dual encounters the tail of a voter arrow in the graphical representation, i.e. if $j \in J(s-)$ and $x = X_{s-}^j$ satisfy $T - s = V_k^x$ for some k such that V_k^x is a single voting time for x. In that case, we set $X_s^j = Y_{x,k}$. The particle coalesces with $X_s^{j'} = Y_{x,k}$ if such a $j' \neq j$ exists, and we remove $j \vee j'$ from $J(s-)$ to form $J(s)$.

To complete the definition of the dual it remains to describe what happens when the dual encounters the tails of $*$-arrows. Let $R_0^{\mathbf{z},T} = 0$ and for $m \geq 1$ let $R_m^{\mathbf{z},T}$ be the first time $s > R_{m-1}^{\mathbf{z},T}$ when a particle in the dual encounters the tail of a $*$-arrow. If

$$j \in J(R_m^{\mathbf{z},T}-) \text{ and } x = X^j_{R_m^{\mathbf{z},T}-} \text{ satisfy } T - R_m^{\mathbf{z},T} = W_{k-1}^x \text{ for some } k \text{ and } |I_k^x| \geq 2, \tag{2.11}$$ mu_mdef

then we set the parent site index $\mu_m = j$. If $|I_k^x| = \ell_m$ and $I_k^x = \{V_{l+1}^x, \dots, V_{l+\ell_m}^x\}$, we create ℓ_m many new particles in the dual by setting $Y_m^i = Y_{x,l+i}$,

$$k(R_m^{\mathbf{z},T}) = k(R_{m-1}^{\mathbf{z},T}) + \ell_m, \text{ and } X^{k(R_{m-1}^{\mathbf{z},T})+i}_{R_m^{\mathbf{z},T}} = Y_m^i, 1 \leq i \leq \ell_m. \tag{2.12}$$ dualdef

Since $|I_k^x| \mid \{|I_k^x| \geq 2\}$ has shifted geometric distribution,

$$P_{G_n,\lambda_n}(\ell_m = k) = \frac{\lambda_n}{(1 + \lambda_n)^{k-1}}, k = 2, 3, \dots \tag{2.13}$$ elldist

The values of the other coordinates remain unchanged, i.e. $X^j_{R_m^{\mathbf{z},T}} = X^j_{R_m^{\mathbf{z},T}-}$ for all $j \in J(R_m^{\mathbf{z},T}-)$. Each 'new' particle immediately coalesces with any particle already present at the vertex where it is born, and we make the necessary changes to $J(R_m^{\mathbf{z},T}-)$ to get $J(R_m^{\mathbf{z},T}) \supseteq J(R_m^{\mathbf{z},T}-)$.

The computation of $\xi_T^{\lambda_n}(z_i)$ using the dual is described in the next subsection. $k(\cdot)$ changes only at the times $\{R_m^{\mathbf{z},T} : m \geq 1\}$, and so it remains unchanged on $[R_m^{\mathbf{z},T}, R_{m+1}^{\mathbf{z},T})$.

Note that $\mathbf{X}_t$ is not measurable with respect to the σ-field generated by the wake up dots and voting times within $[T - t, T]$. Let

$$A_{T,t}^x := \left\{\left|I_{m(x,T-t)}\right| = 1\right\}, \text{ where } m(x, s) := \min\{m \geq 1 : W_m^x > s\}$$

is the last wake up dot for x before time $T - t$ for the time reversed process. Consider the right-continuous filtration $\{\mathcal{F}_t^T : t \geq 0\}$ given by

$$\begin{aligned}\mathcal{F}_t^T := \sigma\big(&\{\Lambda_v^x([T - s, T] \times A) : s \leq t, A \subset \mathcal{N}_x, x \in [n]\} \\ &\cup \{\Lambda_w^x([T - s, T]) : s \leq t, x \in [n]\} \cup \left\{\mathbf{1}_{A_{T,t}^x} : x \in [n]\right\}\big).\end{aligned} \tag{2.14}$$ F_tdef

Then the dual $\mathbf{X}^{\mathbf{z},T}$ is $\mathcal{F}_t^T$-adopted.

mble **Lemma 2.2.** *Let $\{R_m^{\mathbf{z},T}\}$ be the random times defined just before (2.11), $\{(\mu_m, \ell_m, Y_m^1, \dots, Y_m^{\ell_m})\}$ be as defined just after (2.11), and $\{\mathcal{F}_t^T : t \geq 0\}$ be the filtration defined in (2.14). Then*

(1) the dual process $\mathbf{X}_t^{\mathbf{z},T}$ is $\mathcal{F}_t^T$-adopted,

(2) $R_m^{\mathbf{z},T}$ *is* $\mathcal{F}_t^T$*-stopping time and* $R_m^{\mathbf{z},T} \uparrow \infty$ *a.s.*
(3) μ_m, ℓ_m *and* $Y_m^i, 1 \le i \le \ell_m$, *are* $\mathcal{F}^T_{R_m^{\mathbf{z},T}}$ *measurable.*

Proof. Since the wake up dots and the voting times for x are independent Poisson processes with rates λ_n^2 and λ_n respectively, $|I_m^x|, m \in \mathbb{Z}$, are i.i.d. with Geometric$(\lambda_n^2/(\lambda_n+\lambda_n^2))$ distribution so that

$$P_{G_n,\lambda_n}(|I_m^x| = k) = \frac{\lambda_n}{(1+\lambda_n)^{k+1}} \text{ and } P_{G_n,\lambda_n}(|I_m^x| \ge k) = \frac{1}{(1+\lambda_n)^k} \text{ for } k = 0,1,\dots. \quad (2.15)$$ I^x_dist

Specifically, for each $x \in [n]$ and $m \in \mathbb{Z}$, W_m^x is a $*$-dot with probability $1/(1+\lambda_n)^2$ independently of other wake up dots. So if we reverse time, then using the thinning property of the Poisson processes and noting that the time reversed Poisson process is also a Poisson process with same intensity, the $*$-dots for x form an $\mathcal{F}_t^T$-adopted Poisson point process with intensity $\lambda_n^2/(1+\lambda_n)^2$.
(1). Note that the birth of new particles at some time in the dual depends on whether one of the existing particles comes across a $*$-dot or not. Since the $*$-dots by time t are $\mathcal{F}_t^T$-measurable, so are the birth events. On the other hand, jump events depend on whether one of the particles in the dual encounters a single voting time or not. Now if time is reversed, then in order to know whether a voting time $V_k^x \ge T-t$ is single or not we need information about the events $\{\{|I_m^x| = 1\} : m \ge m^x(T,t), x \in [n]\}$. Thus, all the jump events in the dual are $\mathcal{F}_t^T$-adopted, and hence so is $\mathbf{X}$.
(2). Since $\mathbf{X}$ is $\mathcal{F}_t^T$-adopted and $R_m^{\mathbf{z},T}$ is the first time after $R_{m-1}^{\mathbf{z},T}$ that one of the particles in the dual has its first $*$-dot, $R_m^{\mathbf{z},T}$ must be a $\mathcal{F}_t^T$-stopping time by induction on m. Moreover, since at most r new particles are born on every birth event, it is easy to see that $P_{G_n,\lambda_n}(R_{m+1}^{\mathbf{z},T} - R_m^{\mathbf{z},T} > \cdot | \mathcal{F}_{R_m^{\mathbf{z},T}}) \ge P((M+1+rm)^{-1}Z > \cdot)$, where Z has exponential distribution with mean 1. This ensures that $R_m^{\mathbf{z},T} \uparrow \infty$ a.s.
(3). Since μ_m is uniform over $J(R_m^{\mathbf{z},T}-)$, it must be $\mathcal{F}^T_{R_m^{\mathbf{z},T}}$-measurable. By the definition of ℓ_m and Y_m^is, $\ell_m = |I_k^x|$ and $Y_m^i, 1 \le i \le \ell_m$, are chosen uniformly from $\mathcal{N}_x$, where $x = X^{\mu_m}_{R_m^{\mathbf{z},T}-}$ and $T - R_m^{\mathbf{z},T} = W_{k-1}^x$ are as in (2.11). So ℓ_m and Y_m^is are $\mathcal{F}^T_{R_m^{\mathbf{z},T}}$-measurable. ■

sect_zeta

2.4. **The computation process ζ.** Given the coalescing branching random walk $\{\mathbf{X}_s^{\mathbf{z},T} : s \in [0,T]\}$ and a set of initial values $\zeta_0(j) = \xi_0^{\lambda_n}(X_T^j), j \in J(T)$, we will define $\{\zeta_t(i), t \in [0,T], i \le k((T-t)-)\}$ so that on a 'good event'

$$E_T^{\mathbf{z}} \text{ (defined in (2.18)), } , \zeta_t(i) = \xi_t^{\lambda_n}(X_{T-t}^i) \forall t \in [0,T] \text{ and } i \le k((T-t)-). \quad (2.16)$$ duality

Note that $\mathbf{X}$ and ζ have different time directions.

First we complete the initial states by setting $\zeta_0(j) = \zeta_0(i)$ for $j \sim_T i \in J(T)$. The values $\zeta_t(i)$ do not change except at times $t = T - R_k^{\mathbf{z},T}$. So if $h = \max\{m : R_m^{\mathbf{z},T} \le T\}$, then $\zeta_t = \zeta_0$ for $t < T - R_h^{\mathbf{z},T}$. To update the values of ζ at time $T - R_h^{\mathbf{z},T}$ first we consider μ_h and set

$$\zeta_{T-R_h^{\mathbf{z},T}}(\mu_h) = \begin{cases} 1 - \zeta_{(T-R_h^{\mathbf{z},T})-}(\mu_h) \text{ if } \zeta_{(T-R_h^{\mathbf{z},T})-}(\mu_h) \ne \zeta_{(T-R_h^{\mathbf{z},T})-}(Y_h^i) \text{ for at least one } i, \\ \zeta_{(T-R_h^{\mathbf{z},T})-}(\mu_h) \text{ otherwise.} \end{cases}$$

For $k \le k(R_h^{\mathbf{z},T}-)$ and $k \ne \mu_h$, we set

$$\zeta_{T-R_h^{\mathbf{z},T}}(k) = \zeta_{T-R_h^{\mathbf{z},T}}(\mu_h) \text{ if } k \sim_{R_h^{\mathbf{z},T}} \mu_h \text{ and } \zeta_{T-R_h^{\mathbf{z},T}}(k) = \zeta_{(T-R_h^{\mathbf{z},T})-}(k) \text{ otherwise.}$$

The values $\zeta_t(i)$ remain the same for $t \in [T - R_h^{\mathbf{z},T}, T - R_{h-1}^{\mathbf{z},T})$. If $h \geq 2$, we proceed as above. Otherwise we have reached $t = T - R_0^{\mathbf{z},T} = T$, when we set $\zeta_T = \zeta_{T-}$.

Having defined the computation process we now describe $E_T^{\mathbf{z}}$ for which (2.16) holds. Recall the notations $(R_m^{\mathbf{z},T}, x, k, Y_m^i, V_{l+i}^x)$ used in and just after (2.11). The states of Y_m^i at time V_{l+i}^x must be the same as those at time W_{k-1}^x, otherwise there may be a discrepancy between ζ and ξ^{λ_n}. Also whenever an old particle jumps or a new particle is born in the dual process, it will land between two successive wake up dots with high probability. To avoid erroneous computation of states we need to make sure that there is at most one voting time between those two successive wake up dots. Keeping these in mind, we define

$$E_m^{\mathbf{z},T} := \left\{ \xi_t^{\lambda_n}(Y_m^i) = \xi_{W_{k-1}^x}^{\lambda_n}(Y_m^i) \text{ for } t \in [W_{k-1}^x, W_k^x] \text{ and } i \leq |I_k^x| \right\}, \text{ where } (x,k,Y_m^i) \text{ are as in (2.11)},$$

$$N(\mathbf{z},T) := \max\{m \geq 0 : R_m^{\mathbf{z},T} \leq T\}, \text{ and} \tag{2.17}$$

N_T

$$E_T^{\mathbf{z}} := \left(\cap_{m=1}^{N(\mathbf{z},T)} E_m^{\mathbf{z},T}\right) \cap \left(\cap_{\{(x,t): t \leq T, x = X_t^{\mathbf{z},T,i} \neq X_{t-}^{\mathbf{z},T,i}\}} \{W_{m(x,t)-1}^x \text{ is not a } *\text{-dot}\}\right) \tag{2.18}$$

E^T def

dualcond **Lemma 2.3.** *Let $E_T^{\mathbf{z}}$ be as in (2.18). Then (2.16) holds on $E_T^{\mathbf{z}}$ and $P_{G_n,\lambda_n}(E_T^{\mathbf{z}}) = 1 - o(1)$.*

Proof. $E_m^{\mathbf{z},T}$ occurs if there is no voting arrow for the neighbors of x during $[W_{k-1}^x, W_k^x]$. Since x has r neighbors, $P_{G_n,\lambda_n}(E_m^{\mathbf{z},T}) \geq 1 - \lambda_n r/(\lambda_n r + \lambda_n^2)$. This bound and Lemma 2.8 imply

$$\begin{aligned} P_{G_n,\lambda_n}\left(\cap_{m=1}^{N(\mathbf{z},T)} E_m^{\mathbf{z},T}\right) &\geq P_{G_n,\lambda_n}(\cap_{m=1}^{\sqrt{\lambda_n}} E_m^{\mathbf{z},T}) - P_{G_n,\lambda_n}(N(\mathbf{z},T) > \sqrt{\lambda_n}) \\ &\geq 1 - \frac{\sqrt{\lambda_n}}{1 + \lambda_n/r} - \frac{(M+1)e^{rT}}{M+1+2\sqrt{\lambda_n}} = 1 - o(1) \end{aligned}$$

Whenever $x = X_t^i \neq X_{t-}^i$, $W_{m(x,t)-1}^x$ is a $*$-dot with probability $[\lambda_n^2/(\lambda_n + \lambda_n^2)]^2 \leq 1/\lambda_n^2$. Since the expected number of jumps within time $[0,T]$ for each particle is $\leq \lambda_n T + 1$, a single particle encounters such an event with probability $\leq T/\lambda_n + 1/\lambda_n^2$ by Markov inequality. Now if $N(\mathbf{z},T) \leq k$, then the total number of particles is $\leq (M+1) + rk$. So using Lemma 2.8,

$$\begin{aligned} &P_{G_n,\lambda_n}\left(\cup_{\{(x,i,t): 0 \leq t \leq T, i \leq k(t), x = X_t^i \neq X_{t-}^i\}} \{W_{m(x,t)-1}^x \text{ is a } *\text{-dot}\}\right) \\ &\leq P_{G_n,\lambda_n}(N(\mathbf{z},T) > \sqrt{\lambda_n}) + (M+1+r\sqrt{\lambda_n})\left(\frac{T}{\lambda_n} + \frac{1}{\lambda_n^2}\right) = o(1). \end{aligned}$$

Combining the two estinates we get the desired result. ∎

sect_Xhat

2.5. **Branching random walk approximation $\hat{\mathbf{X}}$.** Recall from (2.15) that $P_{G_n,\lambda_n}(|I_m^x| = 1) = \lambda_n/(1+\lambda_n)^2$, which implies that if time is reversed, the rate of the single wake up dots $\{W_m^x : |I_{m+1}^x| = 1\}$ is $\lambda_n^3/(1+\lambda_n)^2$. So when λ_n is large, the random walk steps in the dual are taken very fast (roughly with rate λ_n).

Noting that G_n locally looks like the homogeneous r-tree, random walk on a tree is transient and random walk steps are taken very fast in the dual, the newly born particles will either coalesce quickly among themselves or with the parent within a short time which is $o(n)$, or do not coalesce before time $O(n)$. As in [4], it is difficult to deal with such a process, where births of new particles are quickly followed by coalescence. To avoid this problem we introduce a non-coalescing branching random walk $\hat{\mathbf{X}}$ along with the corresponding computation process $\hat{\zeta}$ which approximates $\mathbf{X}$ and ζ respectively. We will use a coupling between $(\mathbf{X}, \zeta)$ and $(\hat{\mathbf{X}}, \hat{\zeta})$ later to estimate the difference between them.

For $m \geq 1$ let Π_m be the set of all partitions of $\{0, 1, \ldots, m\}$ and we write $i \sim_\pi j$ if i and j are in the same cell of π. For $\pi \in \Pi_m$ let $J_0(\pi)$ be the subset of $\{0, \ldots, m\}$ consisting of the smallest indices of the cells of π and $|\pi| = |J_0(\pi)|$ be the number of cells of π.

For $m \in \{1, \ldots, r\}$ and $\mathbf{Y} = (Y^0, \ldots, Y^m)$ such that $Y^1, \ldots, Y^m \in \mathcal{N}_{Y^0}$, let $\{S_t^{\mathbf{Y}} = (S_t^{\mathbf{Y},0}, \ldots, S_t^{\mathbf{Y},m}), t \geq 0\}$ be the rate one coalescing simple random walk system on the homogeneous r-tree $\mathbb{T}_r$ with paths in $[D([0, \infty), \mathbb{T}_r)]^{m+1}$ and initial state $S_0^{\mathbf{Y}} = \mathbf{Y}$. Since simple random walk on $\mathbb{T}_r$ is transient, for any $t > 0$ we get a random partition of $\{0, \ldots, m\}$ based on the equivalence relation $i \sim^t j$ iff $S_t^{\mathbf{Y},i} = S_t^{\mathbf{Y},j}$. Let $\{a_n\}$ be any sequence such that $a_n \uparrow \infty$ and

$$\Xi_{m,a_n} \text{ be the law on } \Pi_m \text{ associated with the equivalence relation } i \sim^{a_n} j \text{ iff } S_{a_n}^{\mathbf{Y},i} = S_{a_n}^{\mathbf{Y},j}. \tag{2.19}$$ nudef

It is easy to see that

$$\begin{aligned} &\Xi_{m,a_n} \text{ weakly converges to } \Xi_{m,\infty} \text{ as } n \to \infty, \text{ where } \Xi_{m,\infty} \text{ is the law on } \Pi_m \\ &\text{associated with the equivalence relation } i \sim^\infty j \text{ iff } S_t^{\mathbf{Y},i} = S_t^{\mathbf{Y},j} \text{ for some } t > 0. \end{aligned} \tag{2.20}$$

Recalling the distribution of the new particles born during a birth event in the dual $\mathbf{X}$, we let $\mathfrak{M}^l$ be the random size of the subset obtained after l many 'with replacement' draws from $\{1, 2, \ldots, r\}$, *i.e.*, $\mathfrak{M}^l \stackrel{d}{=} |\{L_1, L_2, \ldots, L_l\}|$, where L_is are i.i.d. with common distribution $U_{[r]}$. We will consider the law

$$\Xi_{\mathfrak{M}^\ell, a_n} \text{ on } \cup_{k=1}^r \Pi_k, \text{ where } \ell \text{ has shifted Geometric distribution as in (2.13).} \tag{2.21}$$ nu_ell

Since $\lambda_n \to \infty$ implies $P(\mathfrak{M}^\ell = 1) \to 1/r$ and $P(\mathfrak{M}^\ell = 2) \to 1 - 1/r$, using standard argument for weak convergence

$$\Xi_{\mathfrak{M}^\ell, a_n} \text{ converges weakly to } \Xi_\infty := \frac{1}{r}\Xi_{1,\infty} + \left(1 - \frac{1}{r}\right)\Xi_{2,\infty}. \tag{2.22}$$ nuconv

Fix distinct sites $z_0, \ldots, z_M \in [n]$ and $T > 0$. Our branching random walk $\hat{\mathbf{X}}$ will have paths in $\mathcal{D}$. It will also be associated with a collection of sets of indices $\hat{J}(t) := \{j : \hat{X}_t^j \neq \infty\}, t \geq 0$, and numbers $\hat{k}(t), t \geq 0$ (analogous to $J(t)$ and $k(t)$ of Section 2.3). It will start at time T and will be defined backward in time. Let $\pi_0 \in \Pi_M$ be a partition (defined explicitly in Section 2.7) associated with the initial coalescence in the dual before any birth event. For $k \geq 1$, let $(\ell_k, \mathbf{L}^k, \pi_k)$ be independent of π_0 and i.i.d. such that (i) ℓ_1 has distribution as in (2.13) (ii) $\mathbf{L}^1 = (L_1^1, \ldots, L_{\ell_1}^1)$, where L_i^1s are i.i.d. with common distribution $U_{[r]}$ and (iii) $\pi_1 | (\ell_1, \mathbf{L}^1) \sim \Xi_{\mathfrak{M}^{\ell_1}, \beta\omega_n}$, where $\mathfrak{M}^{\ell_1} = |\{L_1^1, \ldots, L_{\ell_1}^1\}|$, β and ω_n will be explicitly defined in (2.29) and $\Xi_{\cdot,\cdot}$ is defined in (2.19). Based on the sequence of partitions $\{\pi_k, k \geq 0\}$ and $\{(\ell_k, \mathbf{L}^k) : k \geq 1\}$, we define a sequence of subsets of $\mathbb{N}$ inductively as follows:

$$\hat{J}_0 := J_0(\pi_0) \text{ and } \hat{J}_{k+1} := \hat{J}_k \cup \{M + \ell_1 + \cdots + \ell_k + j : j \in J_0(\pi_{k+1}) \setminus \{0\}\} \text{ for } k \geq 0. \tag{2.23}$$ Jhatdef

Similar to the analogues corresponding to $\mathbf{X}$, let $\hat{R}_0^{\mathbf{z},T} = 0$ and conditioned on $\{\ell_m, m \geq 1\}$ and $\{\pi_m, m \geq 0\}$ let $\hat{R}_{m+1}^{\mathbf{z},T} - \hat{R}_m^{\mathbf{z},T}$ be independent and exponentially distributed random variables with mean $[\lambda_n^2 |\hat{J}_m|/(1 + \lambda_n)^2]^{-1}$. Also let $\{\hat{\mu}_m, m \geq 1\}$ be an independent sequence of independent random variables where $\hat{\mu}_m$ is uniform over $\hat{J}_{m-1}$, and set

$$\hat{k}(t) := M + \ell_1 + \cdots + \ell_m \text{ and } \hat{J}(t) := \hat{J}_m \text{ on } \left[\hat{R}_m^{\mathbf{z},T}, \hat{R}_{m+1}^{\mathbf{z},T}\right).$$

In the branching random walk $\hat{\mathbf{X}}$, $\hat{\mu}_m$ is the index of the site which gives birth at time $\hat{R}_m^{\mathbf{z},T}$.

Conditioned on $\{(\hat{R}^{\mathbf{z},T}_m, \hat{\mu}_m, \hat{J}_m) : m \ge 0\}$ we now define $\hat{\mathbf{X}}$ inductively as follows.

$$\hat{\mathbf{X}}^j_0 = \begin{cases} z_j \text{ if } j \in \hat{J}_0 \\ \infty \text{ otherwise.} \end{cases} \tag{2.24}$$ XhatOdef

For $m \ge 0$, the particles $\hat{\mathbf{X}}^j, j \in \hat{J}_m$, follow independent copies of simple random walk (starting from $\hat{\mathbf{X}}^j_{\hat{R}^{\mathbf{z},T}_m}$ respectively) on the time interval $[\hat{R}^{\mathbf{z},T}_m, \hat{R}^{\mathbf{z},T}_{m+1})$. Jumps in the random walk occur when a particle encounters a single voting time. Recalling that θ_x denotes the law of uniform distribution over $\mathcal{N}_x$, at time $\hat{R}^{\mathbf{z},T}_{m+1}$ we set

$$\hat{\mathbf{X}}^j_{\hat{R}^{\mathbf{z},T}_{m+1}} := \begin{cases} \hat{\mathbf{X}}^j_{\hat{R}^{\mathbf{z},T}_{m+1}-} \text{ if } j \in \hat{J}_m \\ \hat{Y}^j_{m+1}, \text{ where } \hat{Y}^j_{m+1} \sim \theta_x \text{ for } x = \hat{\mathbf{X}}^{\hat{\mu}_{m+1}}_{\hat{R}^{\mathbf{z},T}_{m+1}-}, \text{ if } j \in \hat{J}_{m+1} \setminus \hat{J}_m \\ \infty \text{ otherwise.} \end{cases}$$

The choices $\hat{Y}^j_{m+1}$ are made independently. We have set $\hat{J}(t) = \hat{J}_m$ on $[\hat{R}^{\mathbf{z},T}_m, \hat{R}^{\mathbf{z},T}_{m+1})$ so that no coalescence occurs after the birth of the particles. Also note that the number of new particles are less than those in case of $\mathbf{X}$ to mimic the quick coalescence there. Therefore, if we condition on $\{(\ell_m, \mathbf{L}^m, \pi_m)\}$ and time is reversed, then $\hat{\mathbf{X}}$ is a branching random walk, where the initial particles are at $z_j, j \in J_0(\pi_0)$, each particle jumps to a random neighbor whenever it encounters a single voting time and branches at rate $\lambda_n^2/(1+\lambda_n)^2$, and $|\pi_m| - 1$ new particles are born on randomly chosen neighboring vertices of the m-th branching site.

sect_zetahat

2.6. **Computation process $\hat{\zeta}$.** As in Section 2.4, the branching random walk $\{\hat{\mathbf{X}}_s : s \in [0,T]\}$, the associated sequence $\{(\ell_m, \mathbf{L}^m, \pi_m, \hat{R}^{\mathbf{z},T}_m, \hat{\mu}_m)\}$ and a set of initial values $\hat{\zeta}_0(j), j \in \hat{J}(T)$, we now define a computation process $\hat{\zeta}$ for $\hat{\mathbf{X}}$ on the time interval $[0,T]$. We start with the definition of an equivalence relation $\hat{\sim}_{\hat{R}^{\mathbf{z},T}_m}$ on $\{0, 1, \ldots, M + \ell_1 + \cdots + \ell_m\}$.

$$\begin{aligned} &\text{For } 0 \le j, j' \le M, j \hat{\sim}_{\hat{R}^{\mathbf{z},T}_m} j' \text{ iff } j \sim_{\pi_0} j', \\ &\text{for } 1 \le k \le m \text{ and } 1 \le j \le \ell_k, M + \sum_{i=1}^{k-1} \ell_i + j \hat{\sim}_{\hat{R}^{\mathbf{z},T}_m} \hat{\mu}_k \text{ iff } L^k_j \sim_{\pi_k} 0, \\ &\text{for } 1 \le k \le m \text{ and } 1 \le j, j' \le \ell_k, M + \sum_{i=1}^{k-1} \ell_i + j \hat{\sim}_{\hat{R}^{\mathbf{z},T}_m} M + \sum_{i=1}^{k-1} \ell_i + j' \text{ iff } L^k_j \sim_{\pi_k} L^k_{j'}. \end{aligned} \tag{2.25}$$ simhatdef

Extend the definition of the equivalence relation to $[0,T]$ by setting $j \hat{\sim}_t j'$ iff $j \hat{\sim}_{\hat{R}^{\mathbf{z},T}_m} j'$ for $\hat{R}^{\mathbf{z},T}_m \le t < \hat{R}^{\mathbf{z},T}_{m+1}$ and $0 \le j, j' \le M + \ell_1 + \cdots + \ell_m$.

The definition of $\hat{\zeta}$ is similar to that of ζ in Section 2.4 with hats added to the notations. First we set $\hat{\zeta}_0(j') := \hat{\zeta}_0(j)$ for $j' \le \hat{k}(T)$ and $j' \hat{\sim}_T j \in \hat{J}(T)$. The values $\hat{\zeta}_t(j)$ do not change except at times $t = T - \hat{R}^{\mathbf{z},T}_k$. So if $h = \max\{m : \hat{R}^{\mathbf{z},T}_m \le T\}$, then $\hat{\zeta}_t = \hat{\zeta}_0$ for $t < T - \hat{R}^{\mathbf{z},T}_h$.

To update the values of $\hat\zeta$ at time $T-\hat R_h^{\mathbf{z},T}$ we set

$$\hat\zeta_{T-\hat R_h^{\mathbf{z},T}}(\hat\mu_h) = \begin{cases} 1-\hat\zeta_{(T-\hat R_h^{\mathbf{z},T})-}(\hat\mu_h) & \text{if } \hat\zeta_{(T-\hat R_h^{\mathbf{z},T})-}(\hat\mu_h)\neq \hat\zeta_{(T-\hat R_h^{\mathbf{z},T})-}(M+\ell_1+\cdots+\ell_{h-1}+j) \\ & \text{for at least one } j\in\{1,\dots,\ell_h\}, \\ \hat\zeta_{(T-\hat R_h^{\mathbf{z},T})-}(\hat\mu_h) & \text{otherwise.}\end{cases} \tag{2.26}$$

$$\hat\zeta_{T-\hat R_h^{\mathbf{z},T}}(k) = \begin{cases} \hat\zeta_{T-\hat R_h^{\mathbf{z},T}}(\hat\mu_h) & \text{if } k\le \hat k(\hat R_h^{\mathbf{z},T}-) \text{ and } k\neq\hat\mu_h \text{ and } k\sim_{\hat R_h^{\mathbf{z},T}}\hat\mu_h \\ \hat\zeta_{(T-\hat R_h^{\mathbf{z},T})-}(k) & \text{if } k\le \hat k(\hat R_h^{\mathbf{z},T}-) \text{ and } k\not\sim_{R_h^{\mathbf{z},T}}\hat\mu_h.\end{cases} \tag{2.27}$$

The values $\hat\zeta_t(i)$ remain the same for $t\in[T-\hat R_h^{\mathbf{z},T},T-\hat R_{h-1}^{\mathbf{z},T})$. If $h\ge 2$, we proceed as above. Otherwise we have reached $t=T-\hat R_0^{\mathbf{z},T}=T$, when we set $\hat\zeta_T=\hat\zeta_{T-}$.

sect_coupling

2.7. **Coupling of $(\mathbf{X},\zeta)$ and $(\hat{\mathbf{X}},\hat\zeta)$.** Here we describe a construction of the branching random walk $\hat{\mathbf{X}}$ and the associated computation process $\hat\zeta$ using the graphical representation such that if λ_n is large, then with high probability the dual $\mathbf{X}$ is close to $\hat{\mathbf{X}}$ and both ζ and $\hat\zeta$ will compute the same result at time T given identical inputs at time 0. As earlier, fix $z_0,\dots,z_M\in[n]$ and $T>0$. Recall the time-reversed filtration $\mathcal{F}_t^T$ defined in (2.14) and the stopping times $\{R_m^{\mathbf{z},T}: m\ge 1\}$ defined in Section 2.3.

To define the partitions $\{\pi_m : m\ge 0\}$ needed for the construction of $\hat{\mathbf{X}}$, first we introduce the following useful notation. For a $\mathcal{F}_t^T$ stopping time σ and $\mathcal{F}_\sigma^T$ measurable random vector $\mathbf{Y}=(Y_0,\dots,Y_{M'})\in[n]^{M'}$, let $\{\hat S_{\sigma,t}^{\mathbf{Y}}=(\hat S_{\sigma,t}^{\mathbf{Y},0},\dots,\hat S_{\sigma,t}^{\mathbf{Y},M'}) : t\ge\sigma\}$ be a system of coalescing random walks on G_n starting at time σ at locations $\mathbf{Y}$ and satisfying the following jump rule. Whenever a particle in the system encounters the tail of a voting arrow in the graphical representation, it jumps to the other end of it. When $\sigma=0$, we write $\hat S_t^{\mathbf{Y}}$ instead of $\hat S_{0,t}^{\mathbf{Y}}$. For $\sigma,\mathbf{Y}$ as above and any $t>0$, let

$$\pi_{\sigma,\mathbf{Y}}(t)\in\Pi_{M'} \text{ be the random partition at time } \sigma+t \tag{2.28}$$ pi_Y

associated with the equivalence relation $j\sim j'$ iff $\hat S_{\sigma,\sigma+t}^{\mathbf{Y},j}=\hat S_{\sigma,\sigma+t}^{\mathbf{Y},j'}$. Call $\pi_{\sigma,\mathbf{Y}}(t)$ the random partition at time $\sigma+t$ with initial condition $\mathbf{Y}$ at time σ.

In order to have desirable probability estimates for several events we define

$$\omega_n := (1/\varpi)\log_{r-1}(n/\lambda_n) \text{ and } \epsilon_n := \beta\omega_n\lambda_n^{-3}(1+\lambda_n)^2,, \tag{2.29}$$ omega_n

where β and $\varpi=3\varpi_0$ are positive constants defined in Proposition 5.5, and consider the time ϵ_n for the rescaled process $\xi_t^{\lambda_n}$. Since $\log n\ll\lambda_n$ and particles jump roughly at rate $\lambda_n^3/(1+\lambda_n)^2$, the time ϵ_n is small ($\epsilon_n\to 0$ as $n\to\infty$), but for large enough n it corresponds to a large time, which is roughly equal to $\beta\omega_n$ ($\omega_n\to\infty$ as $n\to\infty$), for the unscaled process.

Using the above ingredients and $\mathbf{y}_m := (X_{R_m^{\mathbf{z},T}}^{\mu_m}, Y_m^1,\dots,Y_m^{\ell_m})$ for $m\ge 1$, where $\{\mu_m, R_m^{\mathbf{z},T}, (Y_m^i, 1\le i\le\ell_m)\}$ are as in (2.12), we define

$$\pi_0 := \pi_{0,\mathbf{z}}(\epsilon_n)\in\Pi_M \text{ and for } m\ge 1, \tag{2.30}$$

$$\pi_m := \begin{cases}\pi_{R_m^{\mathbf{z},T},\mathbf{y}_m}(\epsilon_n) & \text{if } R_k^{\mathbf{z},T} > R_{k-1}^{\mathbf{z},T}+\epsilon_n \text{ for all } 1\le k\le m \text{ and } X_{R_m^{\mathbf{z},T}}^{\mu_m}\in L_0(G_n) \\ \pi_m' & \text{otherwise,}\end{cases}$$

where $L_0(G_n)$ is as in (2.3) and $\{\pi_m' : m\ge 1\}$ is an i.i.d. sequence of partitions with law $\Xi_{\mathfrak{M}^\ell,\beta\omega_n}$ ($\Xi_{\mathfrak{M}^\ell,\cdot}$ is defined in (2.21)) and chosen independent of $\mathcal{F}_\infty^T$.

Recall the notations $\mathcal{F}_t^T$ and $m(x,t)$ defined in and just before (2.14) respectively. Also recall the well known facts: (i) time-reversed Poisson processes are Poisson processes with the same intensity; (ii) in case of superposition of two independent Poisson processes (type I and II), the locations of the points are independent of whether the first point is of type I or not; (iii) homogeneous Poisson processes have translation invariance and independent increment properties. Using these facts and (3) of Lemma 2.2, if $\bar{\mathcal{F}}^T_m := \mathcal{F}^T_{R^{\mathbf{z},T}_m+\epsilon_n} \vee \sigma(\pi'_k, k \le m)$, then π_m is $\bar{\mathcal{F}}^T_m$-measurable and is independent of $\bar{\mathcal{F}}^T_{m-1}$.

Now we show that for each $m \ge 1$, π_m has law $\Xi_{\mathfrak{M}^\ell,\beta\omega_n}$, where $\Xi_{\mathfrak{M}^\ell,\cdot}$ is described in (2.21) and $(\beta, \omega_n$ as in (2.29), with high probability. In order to do that, let $\{S^{\lambda_n,\mathbf{y}}_t = (S^{\lambda_n,\mathbf{y},0}_t, \dots, S^{\lambda_n,\mathbf{y},m}_t) : t \ge 0\}$ be a coalescing system of random walks on G_n starting from $S^{\lambda_n,\mathbf{y}}_0 = \mathbf{y} = (y_0, \dots, y_m)$

$$\text{with associated partitions } \pi^{\lambda_n,\mathbf{y}}(t), t > 0, \text{ of } \{0, 1, \dots, m\}, \tag{2.31}$$ pi^lambda

in which each particle jumps at rate $\lambda_n^3/(1+\lambda_n)^2$ to a randomly chosen neighbor.

partitioneq **Proposition 2.4.** *(1) For $y \in [n]$, the random walks $\hat{S}^y_t$ and $S^{\lambda_n,y}_t$ (described just before (2.28) and (2.31) respectively) can be coupled so that*

(a) for any $T, L > 0$ $P_{G_n,\lambda_n}(\sup_{0 \le s \le TS} d(\hat{S}^y_s, S^{\lambda_n,y}_s) \ge L) \le C_{2.4} e^{-L}$,

(b) for any $\varepsilon_n \downarrow 0$, $P_{G_n,\lambda_n}(\sup_{0\le s\le\varepsilon_n} d(\hat{S}^y_s, S^{\lambda_n,y}_s) \ge 2) \le \varepsilon_n + 1/\lambda_n$.

(c) for any $s > 0$ and $k \le r$ if $(\mathfrak{V}^i_s, \dots, \mathfrak{V}^k_s)$ is an ordered tuple of 'without replacement' draws from $\mathcal{N}_{\hat{S}^y_s}$, then

$$d_{TV}\left(\mathcal{L}\left(\mathfrak{V}^i_s\right), U_{[n]}\right) \le d_{TV}\left(\mathcal{L}\left(\hat{S}^y_s\right), U_{[n]}\right) \le d_{TV}\left(\mathcal{L}\left(S^{\lambda_n,y_i}_s\right), U_{[n]}\right), 1 \le i \le k.$$

(2) For any $\mathbf{y} = (y_0, \dots, y_m)$, the coalescing random walk systems $\hat{S}^{\mathbf{y}}_t$ and $S^{\lambda_n,\mathbf{y}}_t$ can be coupled so that

(a) for any $\varepsilon_n \downarrow 0$ the associated partitions $\pi_{0,\mathbf{y}}(\cdot)$ and $\pi^{\lambda_n,\mathbf{y}}(\cdot)$ satisfy $P_{G_n,\lambda_n}(\pi^{\lambda_n,\mathbf{y}}(\varepsilon_n) \neq \pi_{0,\mathbf{y}}(\varepsilon_n)) \le (m+1)(\varepsilon_n + 2/\lambda_n)$.

(b) In addition, if $y_0 \in L_0(G_n)$ (defined in (2.3)) and $y_1, \dots, y_m \in \mathcal{N}_{y_0}$ and $\varepsilon_n \le (1/5)\log_{r-1} n(1+\lambda_n)^2/\lambda_n^3$, then $\pi^{\lambda_n,\mathbf{y}}(\varepsilon_n) \sim \Xi_{m,\varepsilon_n\lambda_n^3/(1+\lambda_n)^2}$ defined just (2.20).

Proof. We couple the locations of two particles as described below so that they follow the random walks $\hat{S}^y_t$ and $S^{\lambda_n,y}_t$ respectively. They start from y at time 0. Whenever the first particle hits the tail of some voter arrow in the time-reversed graphical representation, *i.e.*, it encounters a single voting time, it allocates the jump time (if it is not already allocated) for the second particle to be the corresponding single wake-up dot and jumps to the other side of the voter arrow. The second particle jumps at its allocated jump time to the neighbor of its current location following the trajectory of the first particle. Once the second particle jumps, it waits for the next allocated jump time for jumping again. By the construction of the graphical representation and noting that single wake-up dots occur at rate $\lambda_n^3/(1+\lambda_n)^2$, the two particles have the desired behavior.

It is easy to check that if the first particle jumps k times before the second particle jumps once, then the distance between them after the jump of the second particle is at most $k-1$. To estimate the probability of the above event we use memoryless property of the exponential distribution and the facts that the voting time and the wake-up dots occur at rate λ_n and λ_n^2 respectively. A voting time is followed by a wake-up dot with probability $\lambda_n^2/(\lambda_n+\lambda_n^2)$. So for any $x_1, x_2, \dots \in [n]$ such that $x_{i+1} \in \mathcal{N}_{x_i}, i \ge 1$, the probability that between a single voting time and corresponding single wake-up dot for x_1 there are l or more voting times

$t_2 < \cdots < t_{l+1} < \cdots$ for the vertices $x_2, \ldots, x_{l+1}, \ldots$ respectively is

$$\sum_{k=l}^{\infty} \left(\frac{\lambda_n}{2\lambda_n + \lambda_n^2} \right)^k \frac{\lambda_n^2}{2\lambda_n + \lambda_n^2} \left(\frac{\lambda_n^2}{\lambda_n + \lambda_n^2} \right)^{-1} \leq 1/\lambda_n^l. \qquad (2.32)$$ jumpbd

Hence the distribution of the number of jumps for the first particle between consecutive jumps of the second particle is stochastically dominated by Geometric distribution with mean $\lambda_n/(\lambda_n - 1)$. Consequently, if $N(T)$ is the number of jumps for the second particle by time T and $\mathfrak{G}_1, \mathfrak{G}_2, \ldots$ are i.i.d. an the common law is Geometric with mean $\lambda_n/(\lambda_n - 1)$, then

$$\sup_{0 \leq s \leq T} d\left(\hat{S}_s^y, S_s^{\lambda_n, y}\right) \text{ is stochastically dominated by } \sum_{i=1}^{N(T)+1} \mathfrak{G}_i - N(T). \qquad (2.33)$$ supbd

(1a). Using (2.33) and noting that $E \exp(\sum_{i=1}^{N(T)+1} \mathfrak{G}_i - N(T)) \leq C_{2.4}(T)$ for some constant $C_{2.4}(T)$, the result follows by Markov inequality.
(1b). Using (2.33) and Markov inequality

$$P_{G_n,\lambda_n}\left(\sup_{0 \leq s \leq \varepsilon_n} d\left(\hat{S}_s^y, S_s^{\lambda_n, y}\right) > 1 \right) \leq P\left(\sum_{i=1}^{N(\varepsilon_n)+1} \mathbf{1}_{\{\mathfrak{G}_i > 1\}} \geq 1 \right)$$
$$\leq \; E \sum_{i=1}^{N(\varepsilon_n)+1} \mathbf{1}_{\{\mathfrak{G}_i > 1\}} = E(N(\varepsilon_n) + 1) P(\mathfrak{G}_1 > 1) \leq \frac{\lambda_n \varepsilon_n + 1}{\lambda_n}.$$

(1c) We begin with the second inequality. Let $\mathfrak{J}_s$ be the difference between the number of jumps for the two particles at time s and $\mathfrak{B}_k(v) \subset [n]$ be the set of all vertices which can be reached from v after k random walk steps. The coupling constructed above suggest that

$$\begin{aligned} P_{G_n,\lambda_n}\left(\hat{S}_s^y = v\right) &= \sum_k P_{G_n,\lambda_n}\left(\hat{S}_s^y = v \middle| \mathfrak{J}_s = k\right) P_{G_n,\lambda_n}\left(\mathfrak{J}_s = k\right) \\ &= \sum_k \sum_{u \in \mathfrak{B}_k(v)} P_{G_n,\lambda_n}\left(\hat{S}_s^y = v, S_s^{\lambda_n, y_i} = u \middle| \mathfrak{J}_s = k\right) P_{G_n,\lambda_n}\left(\mathfrak{J}_s = k\right). \end{aligned}$$

Using reversibility of the underlying discrete time simple random walk and by the definition of total variation distance

$$\sum_{u \in \mathfrak{B}_k(v)} P_{G_n,\lambda_n}\left(\hat{S}_s^y = v \middle| S_s^{\lambda_n, y_i} = u, \mathfrak{J}_s = k\right) = 1, |P_{G_n,\lambda_n}(S_s^{\lambda_n, y_i}) - 1/n| \leq d_{TV}(\mathcal{L}(S_s^{\lambda_n, y_i}, U_{[n]})).$$

Combining the above observations with the fact that $S_s^{\lambda_n, y_i}$ and $\mathfrak{J}_s$ are independent we have

$$\begin{aligned} &\left| P_{G_n,\lambda_n}\left(\hat{S}_s^y = v\right) - \frac{1}{n} \right| \\ \leq \; &\sum_k \sum_{u \in \mathfrak{B}_k} \left| P_{G_n,\lambda_n}\left(S_s^{\lambda_n, y_i}\right) - \frac{1}{n} \right| P_{G_n,\lambda_n}(\mathfrak{J}_s = k) P_{G_n,\lambda_n}\left(\hat{S}_s^y = v \middle| S_s^{\lambda_n, y_i} = u, \mathfrak{J}_s = k\right). \end{aligned}$$

Summing both side over $v \in [n]$ and dividing by 2 the second inequality is established.

To prove the first inequality note that for any $u, v \in [n]$ such that $P_{G_n,\lambda_n}(\mathfrak{V}_s^i = u | \hat{S}_s^y = v) = 1/r$.So

$$P_{G_n,\lambda_n}(\mathfrak{V}_s^i = u) = \frac{1}{r} \sum_{v \in \mathcal{N}_u} P_{G_n,\lambda_n}(\hat{S}_s^y = v), \text{ which implies}$$
$$\frac{1}{2} \sum_{u\in[n]} \left| P_{G_n,\lambda_n}(\mathfrak{V}_s^i = u) - \frac{1}{n} \right| \le \frac{1}{2r} \sum_{u\in[n]} \sum_{v\in\mathcal{N}_u} \left| P_{G_n,\lambda_n}(\hat{S}_s^y = v) - \frac{1}{n} \right|.$$

Interchanging the sums over u and v we get the desired inequality.
(2a) We use the coupling between $\hat{S}_t^{y_i}$ and $S_t^{\lambda_n, y_i}$ for $0 \le i \le m$ to define the coupling between the coalescing random walk systems $\hat{S}_t^{\mathbf{y}}$ and $S_t^{\lambda_n,\mathbf{y}}$. It suffices to show that $P_{G_n,\lambda_N}(\hat{S}_{\varepsilon_n}^{\mathbf{y}} \ne S_{\varepsilon_n}^{\lambda_n,\mathbf{y}})$ has the desired upper bound, as $\hat{S}_{\varepsilon_n}^{\mathbf{y}} = S_{\varepsilon_n}^{\lambda_n,\mathbf{y}}$ ensures $\pi_{0,\mathbf{y}}(\varepsilon_n) = \pi^{\lambda_n,\mathbf{y}}(\varepsilon_n)$. If $\sup_{0\le s\le\varepsilon_n} d(\hat{S}_s^{y_i}, S_s^{\lambda_n,y_i}) \le 1$ for all $i = 0, 1, \dots, m$, and if the time ε_n is not between a single voting time and the corresponding wake up dot for the locations of the particles in the $\hat{S}_t^{\mathbf{y}}$ system at that time, *i.e.,* the wake up dot immediately after time (in the time reversed graphical representation) ε_n is not a single wake up dot for each of these locations, then $\hat{S}_{\varepsilon_n}^{\mathbf{y}} = S_{\varepsilon_n}^{\lambda_n,\mathbf{y}}$. Since there are always at most $(m+1)$ particles in the two systems, we can use (1b) to have

$$\begin{aligned} P_{G_n,\lambda_n}(S_{\varepsilon_n}^{\lambda_n,\mathbf{y}} \ne \hat{S}_{\varepsilon_n}^{\mathbf{y}}) &\le (m+1)\left[P_{G_n,\lambda_n}\left(\sup_{0\le s\le\varepsilon_n} d\left(\hat{S}_s^{y_1}, S_s^{\lambda_n,y_1}\right) > 1 \right) + \frac{\lambda_n}{(1+\lambda_n)^2} \right] \\ &\le (m+1)(\varepsilon_n + 2/\lambda_n). \end{aligned}$$

(2b) It follows from the fact that if $y_0 \in L_0(G_n)$ and $y_1, \dots, y_m \in \mathcal{N}_{y_0}$, then $\{S_t^{\lambda_n,\mathbf{y}} : 0 \le t \le (1/5)\log_{r-1} n(1+\lambda_n)^2/\lambda_n^3\}$ has the same distribution as $\{S_{\lambda_n^3 t/(1+\lambda_n)^2}^{\mathbf{y}} : 0 \le t \le (1/5)\log_{r-1} n\}$ (defined in Section 2.5) after appropriate relabeling of the vertices of $\mathbb{T}_r$.. ■

So (2) of Proposition 2.4 ensures that $\{\pi_m : m \ge 1\}$ described in this section and in Section 2.5 have the same distribution with high probability. The next step in the construction of $\hat{\mathbf{X}}$ is to check whether certain 'bad events' occur to it or not. To do so, we use $L_0(G_n), (\epsilon_n, \omega_n)$ and $(\mu_m, R_m^{\mathbf{z},T})$ defined in (2.3), (2.29) and (2.11) to introduce the stopping times

$$\begin{aligned} \tau_m &:= \inf\left\{ s \ge R_{m-1}^{\mathbf{z},T} + \epsilon_n : d(X_s^j, X_s^{j'}) \le \omega_n \text{ for some } j, j' \in J(s) \text{ with } j \ne j' \right\} \\ \sigma_m &:= \inf\{ s \ge R_{m-1}^{\mathbf{z},T} : X_s^j = X_s^{j'} \text{ for some } j \ne j' \text{ with either } j, j' \in J(R_{m-1}^{\mathbf{z},T}-) \text{ or} \\ &\qquad j \in J(R_{m-1}^{\mathbf{z},T}-) \setminus \{\mu_m\} \text{ and } j' \in J(R_{m-1}^{\mathbf{z},T}) \setminus J(R_{m-1}^{\mathbf{z},T}-) \Big\}, \\ \kappa &:= \min\left\{ m : X_{R_m^{\mathbf{z},T}}^{\mu_m} \notin L_0(G_n) \right\}. \end{aligned} \tag{2.34}$$

tausigmakappa

We also let $\mathcal{S}_t^y$ be the continuous time rate one simple random walk on G_n and choose $\varrho > 0$ large enough so that

$$\sup_{y\in[n]} d_{TV}\left(\mathcal{L}(\mathcal{S}_s^y), U_{[n]}\right) \le 1/n^2 \text{ for all } s \ge \varrho \log n. \tag{2.35}$$

varrho

Based on the above stopping times and the choice of ϱ we define the time T_b when one of the four possible bad events occurs.

$$\begin{aligned} T_b \;:=\; & \min\left\{ R_m^{\mathbf{z},T} : m \ge 1 \text{ and either } R_m^{\mathbf{z},T} < R_{m-1}^{\mathbf{z},T} + \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2} \text{ or } \kappa = m \right\} \\ & \wedge \min\left\{ \tau_m : m \ge 1 \text{ and } \tau_m < R_m^{\mathbf{z},T} \right\} \wedge \min\left\{ \sigma_m : m \ge 2 \text{ and } \sigma_m \le R_{m-1}^{\mathbf{z},T} + \epsilon_n \right\}. \end{aligned}$$

We consider the last minima for $m \geq 2$ as we may not have control over the distance between initial particle locations $z_0, \ldots, z_M$. We expect the last two minimum to be large, because after a birth of new particles from the particle μ_m at time $R_m^{\mathbf{z},T}$ we expect some coalescence among the parent particle and its children. After time ϵ_n particles get separated by a distance $O(\omega_n)$ and stay away from each other till the next birth event, when new particles are born in the neighboring sites of a new parent particle and there can again be coalescence of particles only within the new family.

Having defined $\{\pi_m\}$ and T_b, we now construct the branching random walk $\hat{\mathbf{X}}$ on $[0, T_b)$ and $\{(\hat{\mu}_m, \hat{R}_m^{\mathbf{z},T}) : \hat{R}_m^{\mathbf{z},T} \leq T_b\}$ with law as described in Section 2.5. Later we will show that $T_b > T$ with high probability, so that we can define $\hat{\mathbf{X}}$ on $[0, T]$. The coupling of $\mathbf{X}$ and $\hat{\mathbf{X}}$ will be through the definitions of $\{(\pi_m, \mu_m, R_m^{\mathbf{z},T})\}$ and also through the use of the paths of $\mathbf{X}^j$ to define the corresponding paths of $\hat{\mathbf{X}}^j$ for suitable choices of the superscript j as described below.

Our inductive construction begins by setting $\hat{R}_0^{\mathbf{z},T} = 0$. If $R_1^{\mathbf{z},T} \leq \epsilon_n$, then we set $\hat{X}_s^j = X_s^j$ for $j \in J(R_1^{\mathbf{z},T}-)$ and $s \in (0, R_1^{\mathbf{z},T})$, and the construction of $\hat{\mathbf{X}}$ on $[0, T_b)$ is complete as $T_b = R_1^{\mathbf{z},T}$. Otherwise, let $\hat{J}(0) = J_0(\pi_0)$ and define $\hat{\mathbf{X}}_0$ as in (2.24). Observe that

$$\text{if } R_1^{\mathbf{z},T} > \epsilon_n, \text{ then } \hat{J}(0) = J(\epsilon_n) = J_0(\pi_0). \tag{2.36}$$

Jhat0

In that case, for $s \in [0, R_1^{\mathbf{z},T} \wedge T_b)$ we set $\hat{X}_s^j = X_s^j$ for $j \in \hat{J}(0)$. Since $R_1^{\mathbf{z},T} > \epsilon_n$ implies $T_b > \epsilon_n$ and there is no coalescence in $\mathbf{X}$ during $(\epsilon_n, R_1^{\mathbf{z},T} \wedge T_b)$, we can set

$$\hat{J}(s) = \hat{J}(0) \subset J(s) \text{ for } s \in [0, R_1^{\mathbf{z},T} \wedge T_b) \text{ so that } \hat{J}(s) = J(s) = J(\epsilon_n) \text{ for } s \in [\epsilon_n, R_1^{\mathbf{z},T} \wedge T_b).$$

If $R_1^{\mathbf{z},T} < T_b$, then we set $\hat{R}_1^{\mathbf{z},T} = R_1^{\mathbf{z},T}, \hat{\mu}_1 = \mu_1$ and $\hat{J}(R_1^{\mathbf{z},T}) = \hat{J}(0) \cup \{M + j : j \in J_0(\pi_1)\}$. At time $\hat{R}_1^{\mathbf{z},T} = R_1^{\mathbf{z},T}$, we set

$$\hat{X}^j_{\hat{R}_1^{\mathbf{z},T}} = \begin{cases} X^j_{\hat{R}_1^{\mathbf{z},T}-} & \text{if } j \in \hat{J}(0) \\ X^j_{R_1^{\mathbf{z},T}} & \text{if } j \in \hat{J}(R_1^{\mathbf{z},T}) \setminus \hat{J}(0) \\ \infty & \text{otherwise.} \end{cases}$$

Assume now that for some $m \geq 1$, $\hat{\mathbf{X}}$ has been defined on $[0, R_m^{\mathbf{z},T} \wedge T_b)$ with the property that $R_m^{\mathbf{z},T} < T_b$ implies

$$\begin{aligned} &\hat{R}_k^{\mathbf{z},T} = R_k^{\mathbf{z},T}, \hat{\mu}_k = \mu_k, \hat{J}(R_k^{\mathbf{z},T}) = \hat{J}(R_{k-1}^{\mathbf{z},T}) \cup \{M + \ell_1 + \cdots + \ell_{k-1} + j : j \in J_0(\pi_k) \setminus \{0\}\}, \\ &\hat{J}(R_{k-1}^{\mathbf{z},T}) = \hat{J}(s) \subset J(s) \text{ for all } s \in [R_{k-1}^{\mathbf{z},T}, R_k^{\mathbf{z},T}), \text{ and} \\ &\hat{J}(s) = J(s) = J(R_{k-1}^{\mathbf{z},T} + \epsilon_n) \text{ for all } s \in [R_{k-1}^{\mathbf{z},T} + \epsilon_n, R_k^{\mathbf{z},T}) \end{aligned} \tag{2.37}$$

induct hyp

for $1 \leq k \leq m$. The description after (2.36) explains that the above assumption is true for $m = 1$. To extend the definition of $\hat{\mathbf{X}}$ on $[R_m^{\mathbf{z},T} \wedge T_b, R_{m+1}^{\mathbf{z},T} \wedge T_b)$ we may assume that $R_m^{\mathbf{z},T} < T_b$. Then by our assumption, (2.37) holds for all $1 \leq k \leq m$. At time $\hat{R}_m^{\mathbf{z},T} = R_m^{\mathbf{z},T}$ we set

$$\hat{X}^j_{\hat{R}_m^{\mathbf{z},T}} = \hat{X}^j_{\hat{R}_m^{\mathbf{z},T}-} \text{ for } j \in \hat{J}(R_{m-1}^{\mathbf{z},T}) \text{ and } \hat{X}^j_{\hat{R}_m^{\mathbf{z},T}} = X^j_{R_m^{\mathbf{z},T}} \text{ for } j \in \hat{J}(R_m^{\mathbf{z},T}) \setminus \hat{J}(R_{m-1}^{\mathbf{z},T}). \tag{2.38}$$

Xhat_Rm def

For $s \in [R_m^{\mathbf{z},T}, R_{m+1}^{\mathbf{z},T} \wedge T_b)$ and $j \in \hat{J}(R_m^{\mathbf{z},T})$, we set $\hat{J}(s) = \hat{J}(R_m^{\mathbf{z},T})$ and $\hat{X}_s^j = X_s^j$. To verify (2.37) for $k = m + 1$ note that

$$R_{m+1}^{\mathbf{z},T} < T_b \text{ implies } R_m^{\mathbf{z},T} + \epsilon_n < R_{m+1}^{\mathbf{z},T} < T_b, \tag{2.39}$$

imply1

as $R^{\mathbf{z},T}_{m+1} \le R^{\mathbf{z},T}_m + \epsilon_n$ implies $T_b \le R^{\mathbf{z},T}_{m+1}$. So if $R^{\mathbf{z},T}_{m+1} < T_b$, then for all $s \in [R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_m + \epsilon_n]$,

$$\begin{aligned} \hat{J}(s) &= \hat{J}(R^{\mathbf{z},T}_m) = \hat{J}(R^{\mathbf{z},T}_{m-1}) \cup \{M + \ell_1 + \cdots + \ell_{m-1} + j : j \in J_0(\pi_m) \setminus \{0\}\} \\ &= J(R^{\mathbf{z},T}_{m-1} + \epsilon_n) \cup \{M + \ell_1 + \cdots + \ell_{m-1} + j : j \in J_0(\pi_m) \setminus \{0\}\}. \end{aligned} \tag{2.40}$$

The last two equalities follow from (2.37) by using $s = R^{\mathbf{z},T}_{m-1} + \epsilon_n$. $T_b > R^{\mathbf{z},T}_m + \epsilon_n$ also implies $\tau_m \ge R^{\mathbf{z},T}_m$, so that there is no coalescence in $\mathbf{X}$ during $[R^{\mathbf{z},T}_{m-1} + \epsilon_n, R^{\mathbf{z},T}_m)$ making

$$J(R^{\mathbf{z},T}_{m-1} + \epsilon_n) = J(R^{\mathbf{z},T}_m -), \tag{2.41}$$ eq2

and the new particles born at time $R^{\mathbf{z},T}_m$ do not land on existing particles, as no two of the existing particles are neighbors. In addition, $R^{\mathbf{z},T}_{m+1} \wedge T_b > R^{\mathbf{z},T}_m + \epsilon_n$ implies $R^{\mathbf{z},T}_{m+1} \wedge \sigma_{m+1} > R^{\mathbf{z},T}_m + \epsilon_n$ ensuring that there is no birth of new particles during $(R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_m + \epsilon_n]$ and particles that can coalesce during $[R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_m + \epsilon_n]$ are in $\{X^{\mu_m}_{R^{\mathbf{z},T}_m}, Y^1_m, \ldots, Y^{\ell_m}_m\}$. So using the definition of π_m in (2.30) and combining (2.40) and (2.41),

$$\begin{aligned} J(R^{\mathbf{z},T}_m + \epsilon_n) &= J(R^{\mathbf{z},T}_m -) \cup \{M + \ell_1 + \cdots + \ell_{m-1} + j : j \in J_0(\pi_m) \setminus \{0\}\} \\ &= \hat{J}(s) \text{ for all } s \in [R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_m + \epsilon_n]. \end{aligned}$$

Since $\tau_{m+1} \ge R^{\mathbf{z},T}_{m+1} \wedge T_b$, there is no coalescence for $\mathbf{X}$ during $[R^{\mathbf{z},T}_m + \epsilon_n, R^{\mathbf{z},T}_{m+1} \wedge T_b)$ so that $J(s) = J(R^{\mathbf{z},T}_m + \epsilon_n)$ for all $s \in [R^{\mathbf{z},T}_m + \epsilon_n, R^{\mathbf{z},T}_{m+1} \wedge T_b)$. If $T_b > R^{\mathbf{z},T}_{m+1}$, we set $\hat{R}^{\mathbf{z},T}_{m+1} = R^{\mathbf{z},T}_{m+1}, \hat{\mu}_{m+1} = \mu_{m+1}$ and use $k = m+1$ in (2.37) to define $\hat{J}(R^{\mathbf{z},T}_{m+1})$. This completes the description of $\hat{\mathbf{X}}$ on $[0, R^{\mathbf{z},T}_{m+1} \wedge T_b)$ with the property in (2.37) for $1 \le k \le m+1$.

Clearly $\hat{\mu}_{m+1}$ is uniform over $\hat{J}(R^{\mathbf{z},T}_{m+1}-) = J(R^{\mathbf{z},T}_m + \epsilon_n)$ (given π_m), as all the particles present at time $s \ge R^{\mathbf{z},T}_m + \epsilon_n$ are equally likely to be the first to give birth, and it is independent of $\{\hat{\mu}_k : k \le m\}$. Also $\hat{R}^{\mathbf{z},T}_{m+1} - \hat{R}^{\mathbf{z},T}_m$ conditioned on $\mathcal{F}^T_{R^{\mathbf{z},T}_m}$ (and π_m) has an exponential distribution with mean $[\lambda_n^2 |J(R^{\mathbf{z},T}_m + \epsilon_n)|/(1+\lambda_n)^2]^{-1}$. Thus, $\hat{\mathbf{X}}$ behaves like the branching random walk described in Section 2.5 on the interval $[R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_{m+1} \wedge T_b)$.

Since $R^{\mathbf{z},T}_m \uparrow \infty$ as $m \uparrow$, $\hat{\mathbf{X}}$ can be defined by induction on $[0, T_b)$. The above arguments and the inductive proof of (2.37) can be summarized as the following lemma.

F_m **Lemma 2.5.** *For $m \ge 1$, let*

$$\begin{aligned} F_m \;:=\; & \left\{ \wedge_{k=1}^m \left(R^{\mathbf{z},T}_k - R^{\mathbf{z},T}_{k-1} \right) > \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2} \right\} \cap \left\{ \wedge_{k=2}^m \left(\sigma_k - R^{\mathbf{z},T}_{k-1} \right) > \epsilon_n \right\} \\ & \cap \left\{ R^{\mathbf{z},T}_k \le \tau_k \text{ for all } 1 \le k \le m \right\} \cap \{\kappa > m\}. \end{aligned}$$

Then (a) $F_m \subset \{\hat{R}^{\mathbf{z},T}_m = R^{\mathbf{z},T}_m < T_b\}$ and (b) (2.37) holds for all $1 \le k \le m$.

Now we use the above coupling of $\mathbf{X}$ and $\hat{\mathbf{X}}$ to show that with high probability the computation processes ζ and $\hat{\zeta}$ return the same value at time T given identical inputs at time 0. Using the branching times $R^{\mathbf{z},T}_m, m \ge 0$, for the dual $\mathbf{X}^{\mathbf{z},T}$, the events F_m described in Lemma 2.5 and $N(\mathbf{z}, T)$ defined in (2.17) as ingredients, we define

$$F^{\mathbf{z}}_T := F_{N(\mathbf{z},T)+1} \cap \left\{ T \notin \cup_{m=0}^{N(\mathbf{z},T)} [R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_m + \epsilon_n] \right\}. \tag{2.42}$$ F^z_T

The following lemma provides a necessary estimate for the probability of the event $F^{\mathbf{z}}_T$.

F^z_N **Lemma 2.6.** *If $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$, then for any $T > 0$ and $\mathbf{z} = (z_0, \ldots, z_M)$, $\sup_{G+m \in \mathcal{G}_n} P_{G_n,\lambda_n}((F^{\mathbf{z}}_T)^c) = o(1)$.*

Lemma 2.6 suggests that $F_T^{\mathbf{z}}$ occurs with high probability when $G_n \in \mathcal{G}_n$. On this event, $\hat{J}(T) = J(T)$ and both computation processes will compute the same output.

zetacouple **Lemma 2.7.** *On the event $F_T^{\mathbf{z}}$, $\hat{\zeta}_0(j) = \zeta_0(j)$ for all $j \in \hat{J}(T)$ implies $\hat{\zeta}_T(i) = \zeta_T(i)$ for all $i = 0, \dots, M$.*

Specifically, if $\hat{\zeta}_0(j) = \xi_0^{\lambda_n}(X_T^j)$ for all $j \in J(T)$, then Lemma 2.7 and (2.16) suggest that on the event $E_T^{\mathbf{z}} \cap F_T^{\mathbf{z}}$, $\hat{\zeta}_T(i) = \xi_T^{\lambda_n}(z_i)$ for all $i = 0, \dots, M$. This observation will be crucial in proving the Theorem 3.1 below.

Proof of Lemma 2.7. By the definition of $F_{N(\mathbf{z},T)+1} \supset F_T^{\mathbf{z}}$ and Lemma 2.5,

$$\begin{aligned} &R^{\mathbf{z},T}_{N(\mathbf{z},T)} + \epsilon_n < T < R^{\mathbf{z},T}_{N(\mathbf{z},T)+1}, \text{ and so} \\ &\hat{R}^{\mathbf{z},T}_m = R^{\mathbf{z},T}_m, \hat{\mu}_m = \mu_m \text{ for } m \le N(\mathbf{z},T), \text{ and } \hat{k}(s) = k(s) \text{ for } s \in [0,T]. \end{aligned}$$

As mentioned in Section 2.6, the inductive description of $\hat{\zeta}$ is the same as that of ζ with hats added to the relevant notations. In view of the last display, it remains to verify that the equivalence relations $\sim_t$ and $\hat{\sim}_t$ are same. ∎

2.8. **Proof of Lemma 2.6.** We begin with estimating $N(\mathbf{z}, T)$.

N_Tbd **Lemma 2.8.** *Let $\mathbf{z} = (z_0, \dots, z_M)$ and $N(\mathbf{z},T)$ be as in (2.17). Then $P_{G_n,\lambda_n}(N(\mathbf{z},T) > k) \le C_{2.8}(M)\exp(-c_{2.8}(M,T)k)$ for some constants $C_{2.8}, c_{2.8} > 0$.*

Proof. Consider a Yule process which starts with r particles and each particle gives birth to r new particles at rate 1. It is well known that if we let $\mathcal{G}_{r,t}$ be the law of the number of particles in such a Yule process at time t, then $\mathcal{G}_{r,t}$ is r times Geometric with mean e^{rt}. So

$$\mathcal{G}_{r,t}(\{k, k+1, \dots\}) = (1 - e^{-rt})^{k/r}. \tag{2.43}$$ cG_rt

Since each particle in the dual $\mathbf{X}^{\mathbf{z},T}$ give birth at rate ≤ 1 to at most r new particles at a time and there are $(M+1)$ particles in the locations $z_0, \dots, z_M$ at time 0, the total number of particles at time T is stochastically dominated by $\mathfrak{Y}_1 + \dots + \mathfrak{Y}_{\lceil (M+1)/r \rceil}$, where $\mathfrak{Y}_i$s are i.i.d. with common distribution $\mathcal{G}_{r,T}$. Now $N(\mathbf{z},T) > k$ implies that the number of particles at time T is at least $M + 1 + k$, as at least one new particle are born at each birth time. So, using (2.43)

$$\begin{aligned} P_{G_n,\lambda_n}(N(\mathbf{z},T) > k) &\le P\left(\sum_{i=1}^{\lceil (M+1)/r \rceil} \mathfrak{Y}_i > M + 1 + k \right) \\ &\le \lceil (M+1)/r \rceil P\left(\mathfrak{Y}_1 > (M+1+k)/\lceil (M+1)/r \rceil\right) \\ &= \lceil (M+1)/r \rceil \left(1 - e^{-rT}\right)^{(M+1+k)/(r\lceil (M+1)/r \rceil)}. \end{aligned}$$

∎

R_mbd **Lemma 2.9.** *Let $R_m^{\mathbf{z},T}.m \ge 0$, be the $\mathcal{F}_t^T$-stopping times described in Section 2.3 and $N(\mathbf{z},T)$ be as in (2.17) for $\mathbf{z} = (z_0, \dots, z_M)$. Then for ϱ specified in (2.35) and large enough n,*

$$\sup_{G_n \in \mathcal{G}_n} P_{G_n,\lambda_n}\left(\min_{1 \le m \le N(\mathbf{z},T)+1} R_m^{\mathbf{z},T} - R_{m-1}^{\mathbf{z},T} \le \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2} \right) = o(1).$$

Proof. It is easy to see that if Z has exponential distribution with mean 1, then $R^{\mathbf{z},T}_m - R^{\mathbf{z},T}_{m-1}$ stochastically dominates $Z/[M+1+(m-1)r]$ for any $m \geq 1$. This observation together with the inequality $P(Z \leq z) = 1 - e^{-z} \leq z$ and Lemma 2.8 imply that the reqired probability is

$$\begin{aligned}
&\leq \quad P_{G_n,\lambda_n}(N(\mathbf{z},T) \geq k) + \sum_{m=1}^{k} P_{G_n,\lambda_n}\left(\frac{Z}{M+1+(m-1)r} \leq \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2}\right) \\
&\leq \quad C_{2.8}(M)\exp(-c_{2.8}(M,T)k) + \sum_{m=1}^{k}(M+1+(m-1)r)\frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2} \\
&\leq \quad C_{2.8}(M)\exp(-c_{2.8}(M,T)k) + [(M+1)k + rk^2/2]\frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2}
\end{aligned}$$

for any $k \geq 1$. Replacing k by $\lceil [\varrho \log n/(\lambda_n^3/(1+\lambda_n)^2)]^{-1/3} \rceil$ we get the desired bound. ■

taubd **Lemma 2.10.** *If* $\log n\lambda_n \ll n/(\log n)^\eta$ *for some* $\eta > 0$*. Then*

$$\sup_{G_n \in \mathcal{G}_n} P_{G_n,\lambda_n}\left(\max\left\{\tau_m, R^{\mathbf{z},T}_{m-1} + \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2}\right\} \leq R^{\mathbf{z},T}_m \text{ for some } m \leq N(\mathbf{z},T)\right) = o(1).$$

Proof. Observe that for any $m \geq 1$,

$$\begin{aligned}
&P_{G_n,\lambda_n}\left(\tau_m \vee (R^{\mathbf{z},T}_{m-1} + \varrho(1+\lambda_n)^2 \log n/\lambda_n^3) < R^{\mathbf{z},T}_m \middle| \mathcal{F}^T_{R^{\mathbf{z},T}_{m-1}}\right) \\
&\leq P_{G_n,\lambda_n}\Bigg(R^{\mathbf{z},T}_m > R^{\mathbf{z},T}_{m-1} + \varrho(1+\lambda_n)^2 \log n/\lambda_n^3 \text{ and } \exists i,j \in J(R^{\mathbf{z},T}_{m-1} + \epsilon_n), i \neq j, \\
&\qquad \text{such that} \inf_{R^{\mathbf{z},T}_{m-1}+\epsilon_n \leq s \leq R^{\mathbf{z},T}_m} d(X^i_s, X^j_s) \leq \omega_n \Bigg| \mathcal{F}^T_{R^{\mathbf{z},T}_{m-1}}\Bigg).
\end{aligned}$$

Now the condition for i and j in the above expression implies that $i,j \in J(R^{\mathbf{z},T}_{m-1})$ and $X^i_s \neq X^j_s$ for all $s \in [R^{\mathbf{z},T}_{m-1}, R^{\mathbf{z},T}_{m-1} + \epsilon_n]$. So the above is at most

$$\begin{aligned}
\sum_{i,j \in J(R^{\mathbf{z},T}_{m-1}), i \neq j} P_{G_n,\lambda_n}\Bigg(X^i_s \neq X^j_s \forall\, s \in \left[R^{\mathbf{z},T}_{m-1}, R^{\mathbf{z},T}_{m-1} + \epsilon_n\right] \text{ and } d(X^i_s, X^j_s) \leq \omega_n \\
\text{for some } s \in [R^{\mathbf{z},T}_{m-1} + \epsilon_n, T] \Bigg| \mathcal{F}^T_{R^{\mathbf{z},T}_{m-1}}\Bigg). \qquad (2.44)
\end{aligned}$$ taubdstep1

Noting that conditional on $\mathcal{F}^T_{R^{\mathbf{z},T}_{m-1}}$,

$$\left(X^i_{R^{\mathbf{z},T}_{m-1}+s}, X^j_{R^{\mathbf{z},T}_{m-1}+s}\right) \stackrel{d}{=} \hat{S}^{\mathbf{y}}_s, \text{ where } \mathbf{y} = \left(X^i_{R^{\mathbf{z},T}_{m-1}}, X^j_{R^{\mathbf{z},T}_{m-1}}\right), \forall s \in T - R^{\mathbf{z},T}_{m-1}, \qquad (2.45)$$ XScouple

we can use (2) of Proposition 5.5 to bound each of the summands in (2.44) by o(1). Hence the sum in (2.44) is $\leq |J(R^{\mathbf{z},T}_{m-1})|^2 o(1) \leq (M+1+rm)^2 o(1)$. So considering whether $N(\mathbf{z},T) > k$ or no, the probability of interest is

$$\leq P_{G_n,\lambda+n}(N(\mathbf{z},T) > k) + \sum_{m=1}^{k}(M+1+rm)^2 o(1) = P_{G_n,\lambda+n}(N(\mathbf{z},T) > k) + C(M)k^3 o(1).$$

Using Lemma 2.8 and replacing k by a quantity θ_n so that $\theta_n \to \infty$ and $\theta_n o(1) \to 0$ as $n \to \infty$ we get the desired result. ■

sigma_mbd **Lemma 2.11.** *If $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$, then*

$$\sup_{G_n \in \mathcal{G}_n} P_{G_n,\lambda_n}\left(\left\{\sigma_m \le R^{\mathbf{z},T}_{m-1} + \epsilon_n \text{ for some } m \le N(\mathbf{z},T)\right\} \cap \left\{\tau_m > R^{\mathbf{z},T}_m \forall m \le N(\mathbf{z},T)\right\}\right) = o(1).$$

Proof. Note that on the event $\{\tau_{m'} > R^{\mathbf{z},T}_{m'}$ for all $1 \le m' \le m\}$,

$$d\left(X^i_{R^{\mathbf{z},T}_m}, X^j_{R^{\mathbf{z},T}_m}\right) \ge \begin{cases} \omega_n & \forall i,j \in J\left(R^{\mathbf{z},T}_m -\right) \text{ and } i \ne j \\ \omega_n - 1 & \forall i \in J\left(R^{\mathbf{z},T}_m -\right) \setminus \{\mu_m\} \text{ and } j \in J\left(R^{\mathbf{z},T}_m\right) \setminus J\left(R^{\mathbf{z},T}_{m-}\right) \end{cases}$$

In view of (2.45), (1) of Proposition 5.5, and the bound $|J(R^{\mathbf{z},T}_m -)| \le M + 1 + (m-1)$, w have

$$P_{G_n,\lambda_n}\left(\left\{\sigma_{m+1} \le R^{\mathbf{z},T}_m + \epsilon_n\right\} \cap \left\{\tau_{m'} \ge R^{\mathbf{z},T}_{m'} \forall 1 \le m' \le m\right\}\right) \le 2[(M+1+(m-1))^2]o(1).$$

Finally imitating the argument which concludes :emma 2.10 we get the desired result. ∎

kappabd **Lemma 2.12.** *If $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$, then*

$$\sup_{G_n \in \mathcal{G}_n} P_{G_n,\lambda_n}\left(\{\kappa \le N(\mathbf{z},T)\} \cap \left\{\min_{1 \le m \le N(\mathbf{z},T)+1} R^{\mathbf{z},T}_m - R^{\mathbf{z},T}_{m-1} > \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2}\right\}\right) = o(1).$$

Proof. First note that if $R^{\mathbf{z},T}_m > R^{\mathbf{z},T}_{m-1} + \varrho \log n(1+\lambda_n)^2/\lambda_n^3$ and $\kappa = m$, then $X^i_{R^{\mathbf{z},T}_m} \notin L_0(G_n)$ for some $i \in J(R^{\mathbf{z},T}_m -) \subset J(R^{\mathbf{z},T}_{m-1})$. For one such $i \in J(R^{\mathbf{z},T}_m -)$ and $m \le N(\mathbf{z},T)$ if we let $y = X^i_{R^{\mathbf{z},T}_{m-1}}$, then using the couplings in (2.45) and Proposition 2.4

$$\begin{aligned} &P_{G_n,\lambda_n}\left(\left\{X^i_{R^{\mathbf{z},T}_m} \notin L_0(G_n)\right\} \cap \left\{R^{\mathbf{z},T}_m > R^{\mathbf{z},T}_{m-1} + \varrho \log n(1+\lambda_n)^2/\lambda_n^3\right\}\right) \\ &\le P_{G_n,\lambda_n}\left(\sup_{R^{\mathbf{z},T}_{m-1} \le s \le R^{\mathbf{z},T}_m} d\left(\hat{S}^y_s, S^{\lambda_n,y}_s\right) > (1/10)\log_{r-1} n\right) + P_{G_n,\lambda_n}\left(\left\{d\left(S^{\lambda_n,y}_{R^{\mathbf{z},T}_m}, (L_0(G_n))^c\right)\right.\right. \\ &\qquad \left.\left.\text{is at most } (1/10)\log_{r-1} n\right\} \cap \left\{R^{\mathbf{z},T}_m > R^{\mathbf{z},T}_{m-1} + \varrho \log n(1+\lambda_n)^2/\lambda_n^3\right\}\right). \end{aligned} \tag{2.46}$$

kappa break

Using (1b) of Proposition 2.4 the first term in the right hand side of (2.46) is $\le C_{2.4}(T)n^{-3/10}$. To bound the other term recall that

$$S^{\lambda_n,y}_{R^{\mathbf{z},T}_{m-1}+s} \stackrel{d}{=} S^y_{\lambda_n^3 s/(1+\lambda_n)^2}, \text{ so that } d_{TV}(\mathcal{L}(S^{\lambda_n,y}_{R^{\mathbf{z},T}_{m-1}+s}), U_{[n]}) \le 1/n^2 \forall s > \varrho \log n(1+\lambda_n)^2/\lambda_n^3$$

by the choice of ϱ in (2.35). Also by the definition of $L_0(G_n)$, $U_{[n]}(\{v : d(v,(L_0(G_n))^c) \le (1/10)\log_{r-1} n\}) \le C^1_{2.4} n^{4/5} \cdot n^{1/10}/n$. This bound and the TV bound in the last display imply that the second term in the right hand side of (2.46) is $o(1)$. Once again imitating the argument which concludes Lemma 2.10 we get the desired result. ∎

Proof of Lemma 2.6. It is easy to see that the conditional distribution of $R^{\mathbf{z},T}_m - R^{\mathbf{z},T}_{m-1} - s$ given $\{R^{\mathbf{z},T}_m - R^{\mathbf{z},T}_{m-1} > s\}$ stochastically dominates an exponential random variable with mean $\ge (M+1+r(m-1))^{-1}$. This observation and the inequality $1 - e^{-x} \le x$ imply

$$\begin{aligned} &P_{G_n,\lambda_n}\left(T \in \left[R^{\mathbf{z},T}_m, R^{\mathbf{z},T}_m + \epsilon_n\right]\right) \\ &= P_{G_n,\lambda_n}\left(R^{\mathbf{z},T}_m - R^{\mathbf{z},T}_{m-1} \le T - R^{\mathbf{z},T}_{m-1} | R^{\mathbf{z},T}_m - R^{\mathbf{z},T}_{m-1} \ge T - R^{\mathbf{z},T}_{m-1} - \epsilon_n\right) \le (M+1+rm)\epsilon_n. \end{aligned}$$

Using the above estimate and considering whether $N(\mathbf{z},T) > k$ or not,

$$P_{G_n,\lambda_n}\left(T \in \cup_{i=1}^{N(\mathbf{z},T)}\left[R_m^{\mathbf{z},T}, R_m^{\mathbf{z},T} + \epsilon_n\right]\right)$$

$$\le P_{G_n,\lambda_n}(N(\mathbf{z},T) > k) + \sum_{m=1}^{k}(M+1+rm)\epsilon_n \le P_{G_n,\lambda_n}(N(\mathbf{z},T) > k) + C(M)k^2\epsilon_n. \quad (2.47)$$ break up2

Replacing k by $\epsilon_n^{-1/3}$ in the above estimate, using Lemma 2.8, and then combining with Lemma 2.9, 2.10, 2.11 and 2.12 we get the desired result. ∎

3. ODE for the density of the two opinions

ode **Lemma 3.1.** *$u \in C^1([0,\infty))$ satisfies $u'_t = \mathfrak{k}_3 u_t(1-u_t)(1-2u_t)$ and $u_0 = p \in (0,1)$ if and only if for any $\mathfrak{l} > 0$ it satisfies*

$$u_t = \int_0^t \mathfrak{l}e^{-\mathfrak{l}h}\left[(\mathfrak{k}_3/\mathfrak{l})u_h(1-u_h)(1-2u_h) + u_h\right]\, dh + pe^{-\mathfrak{l}t}. \quad (3.1)$$ integeq

The common solutio is given by

$$u_t = \begin{cases} \frac{1}{2}\left[1 + \left(1 + c(p)e^{\mathfrak{k}_3 t}\right)^{-1/2}\right] & \text{if } p \in (1/2,1) \\ p & \text{if } p = 0,1,1/2\,, \text{ where } c(p) = |p-1/2|^{-2} - 1. \\ \frac{1}{2}\left[1 - \left(1 + c(p)e^{\mathfrak{k}_3 t}\right)^{-1/2}\right] & \text{if } p \in (1/2,1) \end{cases} \quad (3.2)$$ usol

Proof. If u_t satisfies (**??**), then changing the variable $w = t - h$

$$u_t = e^{-\mathfrak{l}t}\int_0^t \mathfrak{l}e^{\mathfrak{l}w}\left[(\mathfrak{k}_3/\mathfrak{l})u_w(1-u_w)(1-2u_w) + u_w\right]\, dw + pe^{-\mathfrak{l}t}, \text{ which implies} \quad (3.3)$$ alteq

$$u'_t = e^{-\mathfrak{l}t}\left[\mathfrak{l}e^{\mathfrak{l}t}\left\{(\mathfrak{k}_3/\mathfrak{l})u_t(1-u_t)(1-2u_t) + u_t\right\}\right] - \mathfrak{l}[u_t - pe^{-\mathfrak{l}t}] - p\mathfrak{l}e^{-\mathfrak{l}t} = \mathfrak{k}_3 u_t(1-u_t)(1-2u_t)$$

and $u_0 = p$. Conversely if u_t satisfies the ODE $u'_t = \mathfrak{k}_3 u_t(1-u_t)(1-2u_t)$ and $u_0 = p$, then integrating by parts it is easy to verify that (3.3) holds, and hence (3.1) is satisfied. The solution to the ODE is obtained by the method of partial fractions. ∎

density **Theorem 3.1.** *Suppose $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some constant $\eta > 0$ and $\xi_t^{\lambda_n}$ be the rescaled latent voter model on the random graph G_n having distribution $\tilde{\mathbb{P}}$ such that $\xi_0^{\lambda_n}$ satisfies $(1/n)\sum_{v\in[n]} P_{G_n,\lambda_n}(\xi_0^{\lambda_n}(v) = 0) = p \in (0,1)$. Let $u(\cdot)$ be the solution (as in (3.2)) of the ODE*

$$u'(t) = \mathfrak{k}_3 u(t)(1-u(t))(1-2u(t)), \quad u(0) = p, \quad (3.4)$$ odeeq

where $\mathfrak{k}_3 = \mathfrak{k}_3(r)$ is the probability that three random walks starting from three neighboring vertices of the infinite homogeneous r-tree never hit each other. Then for any fixed $0 < T < \infty$ and $\mathcal{G}_n$ as in (2.4)

$$\sup_{0\le s\le T}\left|\frac{1}{n}\sum_{z\in[n]} P_{G_n,\lambda_n}\left(\xi_s^{\lambda_n}(z) = 0\right) - u(s)\right| \to 0 \text{ as } n \to \infty \text{ uniformly in } G_n \in \mathcal{G}_n.$$

Proof. For $z \in [n]$ and the events E_s^z, F_s^z defined in (2.18) and (2.42) let

$$EF_s^z := E_s^z \cap F_s^z, u_s^{G_n}(z) := P_{G_n,\lambda_n}\left(\xi_s^{\lambda_n}(z) = 0\right), \quad \bar{u}_s^{G_n} := \frac{1}{n}\sum_{z\in[n]} u_s^{G_n}(z).$$

By Lemma 2.3 and 2.6 $\sup_{0<s<T}\sup_{x\in[n]} P_{G_n,\lambda_n}(EF_s^z)=o(1)$. We also need the ingredients

$$p_i^n := \Xi_{\mathfrak{M}^\ell,\beta\omega_n}(|\pi|=i), 1\le i\le r+1, \text{ where } \Xi_{\mathfrak{M}^\ell,\beta\omega_n} \text{ is the law on } \cup_{i=1}^r \Pi_i \tag{3.5}$$ p^n_kdef

as described in Section 2.5, and recall from (2.22) that

$$\mathfrak{k}_i := \lim_{n\to\infty} p_i^n = \Xi_\infty(|\pi|=i) \text{ exists and } \mathfrak{k}_i = \begin{cases} >0 & \text{for } i=1,2,3 \\ =0 & \text{otherwise} \end{cases}. \tag{3.6}$$ fk_i

Now in view of (2.16) and Lemma 2.7 if EF_s^z occurs, then

$$\begin{aligned} &(a)\hat{R}_1^{z,s} > \delta_n := \frac{\varrho \log n}{\lambda_n^3/(1+\lambda_n)^2}, (b) \text{ all branching sites of } \hat{X}_t^{z,s}, t\in[0,s], \text{ are in } L_0(G_n) \\ &(c)\hat{\zeta}_0(j) = \xi_0^{\lambda_n}\left(X_s^{z,s,j}\right) \text{ for all } j\in J(s) \text{ imply } \xi_t^{\lambda_n}(z)=\hat{\zeta}_t(i) \text{ for all } i\in J(s-t), t\in[0,s], \\ &\text{and in particular } \xi_s^{\lambda_n}(z)=\hat{\zeta}_s(0). \end{aligned} \tag{3.7}$$ EFgoodevent

Thinning of Poisson process suggest that if $\mathfrak{R}$ is the first time when new particles are added in $\hat{\mathbf{X}}^{z,s}$, then $\mathfrak{R}$ has exponential distribution with rate $(1-p_1^n)\lambda_n^2/(1+\lambda_n)^2$.

If EF_s^z occurs and $\mathfrak{R}\le s$, then $\mathfrak{R}$ must be in $[\delta_n, s]$ by (a) of (3.7), and k new particle will be born at time $\mathfrak{R}$ with probability $p_{k+1}^n/(1-p_1^n)$ (defined in (3.5)). So using the update rule in (2.26) and the fact that different particles in $\hat{\mathbf{X}}^{z,s}$ move independently,

$$\begin{aligned} &P_{G_n,\lambda_n}\left(\left\{\xi_s^{\lambda_n}(z)=0\right\}\cap EF_s^z\cap\{\mathfrak{R}\in[\delta_n,s]\}\,\middle|\,\mathfrak{R}\right) \\ &= \mathbf{1}_{\{\delta_n\le\mathfrak{R}\le s\}}\sum_{k=1}^r \frac{p_{k+1}^n}{1-p_1^n}\Big[P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(0)=0 \text{ and } \hat{\zeta}_{s-\mathfrak{R}}(i)=0, i=1,\dots,k\right\}\cap EF_s^z\right) \\ &\quad + P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(0)=1 \text{ and } \hat{\zeta}_{s-\mathfrak{R}}(i)=0 \text{ for at least one} i=1,\dots,k\right\}\cap EF_s^z\right)\Big] \\ &= \mathbf{1}_{\{\delta_n\le\mathfrak{R}\le s\}}\sum_{k=1}^r \frac{p_{k+1}^n}{1-p_1^n}\Big[P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(0)=0\right\}\cap EF_s^z\right)\cdot \\ &\qquad \prod_{i=1}^k P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(i)=0\right\}\cap EF_s^z\right) + P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(0)=1\right\}\cap EF_s^z\right)\cdot \\ &\qquad \left\{1-\prod_{i=1}^k P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(i)=1\right\}\cap EF_s^z\right)\right\}\Big] + o(1). \end{aligned} \tag{3.8}$$ fRbreakup

The $o(1)$ term arises because of ignoring some power of $P_{G_n,\lambda_n}(EF_s^z)$ which would appear in the last step above. On the event $EF_s^z\cap\{\mathfrak{R}<s\}$, $\hat{\zeta}_{s-\mathfrak{R}}(i)=\xi_{s-\mathfrak{R}}^{\lambda_n}(\hat{X}_{\mathfrak{R}}^{z,s,i})$ by (3.7). Also the branching site $\hat{X}_{\mathfrak{R}}^{z,s,0}\overset{d}{=}\hat{S}_{\mathfrak{R}}^z$ and the locations of the new particles $\hat{X}_{\mathfrak{R}}^{z,s,i}\overset{d}{=}\mathfrak{V}_{\mathfrak{R}}^i$ described in (1c) of Proposition 2.4. By our choice of ϱ in (2.35), $d_{TV}(\mathcal{L}(\hat{S}_{\mathfrak{R}}^z),U_{[n]})\le 1/n^2$ on $\{\mathfrak{R}\ge\delta_n\}$. Combining the last three observations and using (1c) of Proposition 2.4

$$\begin{aligned} &\left|P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(0)=0\right\}\cap EF_s^z\right)-\bar{u}_{s-\mathfrak{R}}^{G_n}\right| = \left|P_{G_n,\lambda_n}\left(\left\{\xi_{s-\mathfrak{R}}^{\lambda_n}(\hat{S}_{\mathfrak{R}}^z)=0\right\}\cap EF_s^z\right)-\bar{u}_{s-\mathfrak{R}}^{G_n}\right| \\ &= \left|P_{G_n,\lambda_n}\left(\xi_{s-\mathfrak{R}}^{\lambda_n}(\hat{S}_{\mathfrak{R}}^z)=0\right)-\bar{u}_{s-\mathfrak{R}}^{G_n}\right| + o(1) \le d_{TV}\left(\mathcal{L}\left(\hat{S}_{\mathfrak{R}}^z\right), U_{[n]}\right)+o(1)=o(1) \end{aligned}$$

on the event $\{\mathfrak{R} \geq \delta_n\}$. Similarly for any $i \in \{1, \ldots, r\}$, $\left|P_{G_n,\lambda_n}\left(\left\{\hat{\zeta}_{s-\mathfrak{R}}(i) = 0\right\} \cap EF_s^z\right) - \bar{u}^{G_n}_{s-\mathfrak{R}}\right| = o(1)$ on the event $\{\mathfrak{R} \geq \delta_n\}$. Hence we can rewrite the right hand side of (3.8) to have

$$\begin{aligned} &P_{G_n,\lambda_n}\left(\{\xi_s^{\lambda_n}(z) = 0\} \cap EF_s^z \cap \{\mathfrak{R} \in [\delta_n, s]\} \middle| \mathfrak{R}\right) \\ &= \mathbf{1}_{\{\delta_n \leq \mathfrak{R} \leq s\}} \textstyle\sum_{k=1}^r \frac{p^n_{k+1}}{1-p^n_1} \left[\left(\bar{u}^{G_n}_{s-\mathfrak{R}}\right)^{k+1} + \left(1 - \bar{u}^{G_n}_{s-\mathfrak{R}}\right)\left\{1 - \left(1 - \bar{u}^{G_n}_{s-\mathfrak{R}}\right)^k\right\}\right] + o(1). \quad (3.9) \end{aligned}$$

On the other hand if EF_2^z occurs and $\mathfrak{R} > s$, then using independence of $\xi_0^{\lambda_n}$ and the ingredients for the graphical representation for the interval $(0, s]$,

$$P_{G_n,\lambda_n}\left(\left\{\xi_s^{\lambda_n}(z) = 0\right\} \cap EF_s^z \cap \{\mathfrak{R} > s\}\right) = P(\xi_0^{\lambda_n}(\hat{X}_s^{z,s,0}) = 0) P_{G_n,\lambda_n}\left(\cap EF_s^z \cap \{\mathfrak{R} > s\}\right).$$

Recalling that $d_{TV}(\mathcal{L}(X_t^{z,s,0}), U_{[n]}) \leq 1/n^2$ for all $t \geq \delta_n$, (a) of (3.7) and the property of $\xi_0^{\lambda_n}$ in our hypothesis suggest that the above is

$$(p + o(1)) P_{G_n,\lambda_n}\left(EF_s^z \cap \{\mathfrak{R} > s\}\right) = p\exp(-(1 - p_1^n)s) + o(1). \quad (3.10)$$ nohit

Putting (3.6), (3.9) and (3.10) together, using the distribution of $\mathfrak{R}$ and noting that $\delta_n \to 0$,

$$\begin{aligned} u_s^{G_n}(z) &= \int_{\delta_n}^s (1 - p_1^n)\exp(-(1 - p_1^n)s') \sum_{k=1}^r \frac{p^n_{k+1}}{1 - p_1^n}\Big[(\bar{u}^{G_n}_{s-s'})^{k+1} \\ &\qquad + (1 - \bar{u}^{G_n}_{s-s'})\{1 - (1 - \bar{u}^{G_n}_{s-s'})^k\}\Big]\, ds' + p\exp(-(1 - p^n)s) + o(1) \\ &= \int_0^s (1 - \mathfrak{k}_1)e^{-(1-\mathfrak{k}_1)s'}\left[\frac{\mathfrak{k}_2}{1 - \mathfrak{k}_1}\bar{u}^{G_n}_{s-s'} + \frac{\mathfrak{k}_3}{1 - \mathfrak{k}_1}\left\{\left(\bar{u}^{G_n}_{s-s'}\right)^3\right.\right. \\ &\qquad \left.\left. + \left(1 - \bar{u}^{G_n}_{s-s'}\right)\left(2\bar{u}^{G_n}_{s-s'} - \left(\bar{u}^{G_n}_{s-s'}\right)^2\right)\right\}\right] ds' + pe^{-(1-\mathfrak{k}_1)s} + o(1). \end{aligned}$$

As the $o(1)$ term above doesn't depend on z, we can replace $u_s^{G_n}(z)$ by $\bar{u}_s^{G_n}$ in the above equality. Since $\mathfrak{k}_1 + \mathfrak{k}_2 + \mathfrak{k}_3 = 1$, we write $(1 - \mathfrak{k}_3/(1 - \mathfrak{k}_1))$ in place of $\mathfrak{k}_2/(1 - \mathfrak{k}_1)$ and do a little arithmetic to conclude

$$\begin{aligned} \bar{u}_s^{G_n} &= \int_0^s (1 - \mathfrak{k}_1)e^{-(1-\mathfrak{k}_1)s'}\left[\frac{\mathfrak{k}_3}{1 - \mathfrak{k}_1}\bar{u}^{G_n}_{s-s'}\left(1 - \bar{u}^{G_n}_{s-s'}\right)\left(1 - 2\bar{u}^{G_n}_{s-s'}\right) + \bar{u}^{G_n}_{s-s'}\right] ds' \\ &\quad + pe^{-(1-\mathfrak{k}_1)s} + o(1). \end{aligned}$$

Now if u_t satisfies (3.4), then Lemma 3.1 suggests that u_t also satisfies

$$u_s = \int_0^s (1 - \mathfrak{k}_1)e^{-(1-\mathfrak{k}_1)s'}\left[\frac{\mathfrak{k}_3}{1 - \mathfrak{k}_1}u_{s-s'}\left(1 - u_{s-s'}\right)\left(1 - 2u_{s-s'}\right) + u_{s-s'}\right] ds' + pe^{-(1-\mathfrak{k}_1)s}.$$

Combining last two displays and noting that the Lipschitz constant for the polynomial $(\mathfrak{k}_3/(1-\mathfrak{k}_1))u(1 - u)(1 - 2u) + u$ on the interval $[0, 1]$ is $1 + \mathfrak{k}_3/(1 - \mathfrak{k}_1)$,

$$\left|\bar{u}_s^{G_n} - u_s\right| \leq o(1) + \left(1 + \frac{\mathfrak{k}_3}{1 - \mathfrak{k}_1}\right)\int_0^s \left|\bar{u}^{G_n}_{s-h} - u_{s-h}\right|(1 - \mathfrak{k}_1)e^{-(1-\mathfrak{k}_1)h}\, dh$$

Using standard argument (*e.g.*,Lemma 3.3 in [1]) the above implies

$$\left|\bar{u}_s^{G_n} - u_s\right| \leq o(1)\sum_{k=0}^\infty F^{*k}(s) = o(1)e^{\mathfrak{k}_3 s}, \text{ where}$$

F^{*k} is k^{th} convolution of $F(s) = (1 - \mathfrak{k}_1 + \mathfrak{k}_3)e^{-(1-\mathfrak{k}_1)s}\mathbf{1}_{\{s>0\}}$. Finally noting that the $o(1)$ term works for any $s \in [0, T]$, the desired result follows. ■

4. Lower bound for consensus time

In order to infer about the consensus time we need to estimate the correlation among the states of different vertices. In this section, our goal is to establish the fact that dual processes starting from distant vertices do not collide with high probability, so that their states are asymptotically uncorrelated. We call the particles in the dual process $\mathbf{X}^{y,T}$ members of a y-family.

allcolide1 **Proposition 4.1.** *Let $G_n \in \mathcal{G}_n$ and $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$. There is a constant $\varpi_0 > 0$ (as in Proposition 5.5) such that if $1 < K < K'$ and $y_i \in [n], i \in [K']$, satisfy $d(y_i, y_j) \geq (1/\varpi_0)\log_{r-1}(n/\lambda_n)$ for all $i \in [K], j \in [K']$ and $i \neq j$, then for any fixed $a, T > 0$,*

$$P_{G_n,\lambda_n}\left(\cap_{i\in[K]} \cup_{j\in[K'],j\neq i} \{y_i\text{-family comes within distance } a \text{ of } y_j\text{-family before time } T\}\right)$$
$$\leq c_{4.1}(n/\lambda_n)^{-\sqrt{K}\rho_{4.1}/\varpi_0} \text{ for some positive constants } c_{4.1} = c_{4.1}(K,a), \rho_{4.1} = \rho_{4.1}(\eta, T).$$

Proof. We prove the result for $a = 0$ only, as the other cases are similar. For notational convenience, we also assume without loss of generality that $y_i = i$.

Let ϖ_o, ρ be as in Proposition 5.5 and $\omega_n = (1/\varpi_0)\log_{r-1}(n/\lambda_n)$. We write $\mathbf{K}'$ for $[K']\setminus[K]$ and associate a forest $\mathbb{F}$ on the node set $[K]\cup\{\mathbf{K}'\}$ with the coalescence structure of $\mathbf{X}_t^{[K']}, 0 \leq t \leq T$ as follows. Whenever i-family collides with j-family for some $1 \leq i < j \leq K$ or $j = \mathbf{K}'$, we put an oriented edge $j \twoheadrightarrow i$ in $\mathbb{F}$, merge the two families and declare i to be the family head. A node with no outgoing edge is the root of the component containing it. The event of our interest implies that number of components in $\mathbb{F}$ is at most $K/2 + 1$. We will present the proof in two steps.

Step 1. First we will show that for any distinct f_is

$$P_{G_n,\lambda_n}(f_1, \ldots, f_k \twoheadrightarrow 1 \text{ in } \mathbb{F}) \leq (r-1)^{-k\rho\omega_n/12} \text{ for some } \rho > 0 \tag{4.1}$$ collidebd3

and for large enough n. For $\mathbf{s} = (s_1, \ldots, s_k)$ let $H(\mathbf{s})$ denotes the joint distribution of the hitting times of the families with family heads $1, f_1, \ldots, f_k$ and let $\mathbf{w} = (w_1, \ldots, w_k) \in [n]^k$ be the locations of the coalescence. Also let $\mathfrak{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_k)$ be the permutation of $(f_1, \ldots, f_k)$ corresponding to the order in which coalescence occurs and $\mathfrak{l}_j \in J^{\mathfrak{f}_j}(s_j-), \mathfrak{m}_j \in J^1(s_j), 1 \leq j \leq k$, be the (random) indices of the members of the two families respectively which coalesce at time s_j. Then

$$P_{G_n,\lambda_n}(f_1, f_2, \ldots f_k \twoheadrightarrow 1 \text{ in } \mathbb{F}) \leq \sum_{\mathfrak{f}\in Perm(\mathbf{f})} \int_{0<s_1<\cdots<s_k<T} H(d\mathbf{s}) \sum_{\mathbf{w}\in[n]^k} P_{G_n,\lambda_n}\Big(X_{s_j}^{\mathfrak{f}_j,T,\mathfrak{l}_j} = w_j$$
$$= X_{s_j}^{1,T,\mathfrak{m}_j} \text{ and } X_s^{\mathfrak{f}_j,T,l} \neq X_s^{i,T,l'} \text{ if } \mathfrak{f}_j \neq i, s < s_j\Big). \tag{4.2}$$ col break

We write $D_j = D(\mathfrak{f}_j, \omega_n/6)$, divide $[n]^k$ into 2^k groups depending on whether $w_j \in D_j$ or not and use independence until collision' property of the dual processes to bound the inner sum in the last display by

$$\sum_{\mathbf{b}\in\{0,1\}^k} \sum_{\mathbf{w}:w_j\in(1-b_j)D_j+b_jD_j^c} P_{G_n,\lambda_n}\left(\cap_{j=1}^k\left\{X_{s_j}^{1,T,\mathfrak{m}_j} = w_j\right\}\right)\prod_{j=1}^k P_{G_n,\lambda_n}\left(X_{s_j}^{\mathfrak{f}_j,T,\mathfrak{l}_j} = w_j\right). \tag{4.3}$$ collidebd1

If $w_j \in D_j^c$, then Lemma 2.8 and the second assertion of (1) of Proposition 5.5 imply that for a suitable choice of c

$$P_{G_n,\lambda_n}\left(X_s^{\mathfrak{f}_j,T,\mathfrak{l}_j} = w_j\right) \leq P_{G_n,\lambda_n}(|J^{\mathfrak{f}_j}(s_j)| > c\omega_n) + c\omega_n(r-1)^{-\rho\omega_n/6} \leq (r-1)^{-\rho\omega_n/8}.$$

Using this bound, writing $|\mathbf{b}| = \sum_j b_j$ and summing over all $w_j \in D_j^c$ such that $b_j = 1$ the inner sum in (4.3) is

$$\leq (r-1)^{-|\mathbf{b}|\rho\omega_n/8} \sum_{w_j \in D_j : b_j = 0} \prod_{j : b_j = 0} P_{G_n,\lambda_n}\left(X_{s_j}^{\mathfrak{f}_j, T, \mathfrak{l}_j} = w_j\right) P_{G_n,\lambda_n}\left(\cap_{j:b_j=0}\left\{X_{s_j}^{1,T,\mathfrak{m}_j} = w_j\right\}\right) \tag{4.4}$$

collidebd2

Now we want to estimate the last term of (4.4). It is easy to see that the particle $X_{s_j}^{1,T,\mathfrak{m}_j}$ originates from $\{1, \mathfrak{f}_1, \ldots, \mathfrak{f}_{j-1}\}$. $d(\mathfrak{f}_j, \mathfrak{f}_i), d(\mathfrak{f}_j, 1) \geq \omega_n$ for all $i < j$. For any choice of vertices $w_j \in D_j, j \in \{j : b_j = 0\}$, and indices $m_j, j \in \{j : b_j = 0\}$, $X_{s_j}^{1,T,m_j} = w_j$ implies either starting above $\omega_n/3$ the distance between $\mathfrak{f}_j$ and the particle at $X_{s_j}^{1,T,m_j}$ reduces to $\omega_n/6$ (when the particle is born outside $D(\mathfrak{f}_j, \omega_n/3)$), or starting above $2\omega_n/3$ the distance between $\mathfrak{f}_j$ and the parent (with index m_i for some $i < j$) of the particle at $X_{s_j}^{1,T,m_j}$ reduces to $\omega_n/3$. Hence the Markov property of P_{G_n,λ_n} and repeated application of the second assertion of Proposition 5.5 imply that for any $(w_j \in D_j : b_j = 0)$, $P_{G_n,\lambda_n}\left(\cap_{j:b_j=0}\left\{X_{s_j}^{1,T,m_j} = w_j\right\}\right) \leq (r-1)^{-(k-|b|)\rho\omega_n/3}$. Combining this with the fact that the number of possible values of $\mathfrak{m}_j$ is $\leq j \cdot c\omega_n$. on the event $\{|J_{s_j}^{\mathfrak{f}_j}| \leq c\omega_n\}$ and using Lemma 2.8, the last term of (4.4) is

$$\begin{aligned} P_{G_n,\lambda_n}\left(\cap_{j:b_j=0}\left\{X_{s_j}^{1,T,\mathfrak{m}_j} = w_j\right\}\right) \quad &\leq \quad P_{G_n,\lambda_n}\left(\cup_{j=1}^k \left\{|J^{f_j}(s_j-)| > c\omega_n\right\}\right) \\ &\quad + \prod_{j:b_j=0} (jc\omega_n)(r-1)^{-(k-|b|)\rho\omega_n/3} \leq (r-1)^{-(k-|b|)\rho\omega_n/8}. \end{aligned}$$

for a suitable choice of c. Using this bound the expression in (4.4) is $\leq (r-1)^{-k\rho\omega_n/8}$. Therefore, considering all possible choices of $\mathbf{b}$ in (4.3) and $\mathfrak{f}$ in (4.2) we get

$$P_{G_n,\lambda_n}(f_1, f_2, \ldots, f_k \twoheadrightarrow 1 \text{ in } \mathbb{F}\ ) \leq k!2^k(r-1)^{-k\rho\omega_n/8} \leq (r-1)^{-k\rho\omega_n/12} \tag{4.5}$$

allto1

for large enough n.

Step 2. The bound in (4.1) and the 'independence until coalescence' property of the dual processes imply that if i and j are either from different components or from same component but have equal oriented distance from the root, then

$$P_{G_n,\lambda_n}(\{i_1, \ldots, i_k \twoheadrightarrow i \text{ in } \mathbb{F}\ \} \cap \{j_1, \ldots, j_l \twoheadrightarrow j \text{ in } \mathbb{F}\ \}) \leq (r-1)^{-(k+l)\rho\omega_n/12}. \tag{4.6}$$

collidebd4

Also if $f_1 \twoheadrightarrow f_2 \twoheadrightarrow \cdots \twoheadrightarrow f_k \twoheadrightarrow 1$ is an oriented path in $\mathbb{F}$, then we can interchange the roles of the indices 1 and f_1 to have $f_1 \twoheadrightarrow 1, \ldots, f_k \twoheadrightarrow 1$ in the new labeling. Thus

$$P_{G_n,\lambda_n}(f_1 \twoheadrightarrow 1, \ldots, f_k \twoheadrightarrow 1 \text{ in } \mathbb{F}) \leq (r-1)^{-k\rho\omega_n/12}. \tag{4.7}$$

collidebd5

Now note that if the tree component containing 1 is $\mathfrak{F}$, then either there are at least $\lceil\sqrt{|\mathfrak{F}|-1}\rceil$ nodes in $\mathfrak{F}$ at equal distance from the root or there is an oriented path in $\mathfrak{F}$ of length at least $\lceil\sqrt{|\mathfrak{F}|-1}\rceil$. So using the bounds in (4.6) and (4.7)

$$P_{G_n,\lambda_n}(\mathfrak{F} \text{ is the component containing } 1) \leq 2(r-1)^{-\sqrt{|\mathfrak{F}|-1}\rho\omega_n/12}.$$

The above estimate together with (4.6) and the inequality $\sum_i \sqrt{x_i} > \sqrt{\sum_i x_i}$ implies

$$\begin{aligned} &P_{G_n,\lambda_n}(\mathfrak{F}_1, \ldots, \mathfrak{F}_L \text{ are the tree components of } \mathbb{F}) \\ &\leq 2(r-1)^{-\sum_{i=1}^L \sqrt{|\mathfrak{F}_i|-1}\rho\omega_n/12} \leq 2(r-1)^{-\sqrt{K+1-L}\rho\omega_n/12}, \text{ which in turn implies} \\ &P_{G_n,\lambda_n}(\mathbb{F} \text{ has at most } K/2+1 \text{ components}) \leq c(K)(r-1)^{-\sqrt{(K+1)-(K/2+1)}\rho\omega_n/12}, \end{aligned}$$

where $c(K)$ is twice the number of forests on the node set $[K]\cup\{\mathbf{K}'\}$ with at most $K/2+1$ components. This completes the proof. ∎

Next we use the estimate in Proposition 4.1 to obtain the following large deviation estimate for the number of individuals with opinion 0 at time T.

momentbd **Lemma 4.2.** *Let $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta>0$ and $\boldsymbol{\xi}_t^{\lambda_n}=\{v\in[n]:\xi_t^{\lambda_n}(v)=0\}$. Then for any $k\ge 1$ and $\delta>0$ there are constants $C_{4.2}(k,\delta),\rho_{4.2}(\eta,T)$ so that for large enough n,*

$$P_{G_n,\lambda_n}\left(\sup_{0\le s\le T}\Big||\boldsymbol{\xi}_s^{\lambda_n}|-E_{G_n,\lambda_n}|\boldsymbol{\xi}_s^{\lambda_n}|\Big|>\delta n\right)\le C_{4.2}(n/\lambda_n)^{-\sqrt{k}\rho_{4.2}}$$

Proof. Let ω_n be as in the proof of Proposition 4.1. For $\mathcal{I}\subset[n]$, we say that $i\in\mathcal{I}$ is a 'good element' of $\mathcal{I}$ if $\min_{i\neq i'\in\mathcal{I}} d(i,i')$ is not less than ω_n, otherwise we call it a 'bad element' for $\mathcal{I}$. Define

$$W:=\{(i_1,\dots,i_{2k})\in[n]^{2k}:\text{ at least } k \text{ many indices of } \{i_1,\dots,i_{2k}\} \text{ are good for it}\}.$$

Now let $Y_{v,s}$ be the indicator of the event $\{\xi_s^{\lambda_n}(v)=0\}$ minus its mean under P_{G_n,λ_n} and $U_{n,s}:=\sum_{v\in[n]}Y_{v,s}$. We estimate the even moments of U_n. Noting that $U_n^{2k}=\sum_{i_1,\dots,i_{2k}\in[n]}Y_{i_1,s}\cdots Y_{i_{2k},s}$ and $|Y_{i_j,s}|\le 1$,

$$E_{G_n,\lambda_n}U_{n,s}^{2k}\le|W^c|+\sum_{i_1,\dots,i_{2k}\in W}E_{G_n,\lambda_n}\left[Y_{i_1,s}\cdots Y_{i_{2k},s}\right]. \tag{4.8}$$ moment break

To bound $|W^c|$ note that $|W^c|=\sum_{l=0}^{k-1}|W_l|$, where $W_l:=\{(i_1,\dots,i_{2k}):\{i_1,\dots,i_{2k}\}\text{ has } l \text{ good indices}\}$. To bound $|W_l|$ observe that l many good indices can be chosen in at most n^l ways and $2k-l$ bad indices can be chosen in at most $[n\cdot r(r-1)^{\omega_n-1}]^{(2k-l)/2}$ ways, as the worst situation is to have $(2k-l)/2$ pairs of indices such that the distance between two vertices of any pair is at most ω_n. Since the number of permutations of the elements in $\{i_1,\dots,i_{2k}\}$ is at most $(2k)!$,

$$|W^c|\le(2k)!\sum_{l=0}^{k-1}n^{l+(2k-l)(1+1/\varpi_0)/2}\le k\cdot(2k)!n^{k(3/2+1/2\varpi_0)}. \tag{4.9}$$ bad summands b

The next step is to bound the summands in (4.8) for $(i_1,\dots,i_{2k})\in W$. For $\mathbf{i}=(i_1,\dots,i_{2k})\in W$ we assume (without loss of generality) that $i_1,\dots,i_k$ are good elements for $\{i_1,\dots,i_{2k}\}$, write $\mathbf{i}':=\{i_{k+1},\dots,i_{2k}\}$ and $i_1\prec\cdots\prec i_k\prec\mathbf{i}'$. We associate independent copies of the graphical representation to $i_1,\dots,i_k,\mathbf{i}'$. Based on them we construct $\tilde{\mathbf{X}}^{i_1,T},\dots,\tilde{\mathbf{X}}^{i_k,T},\tilde{\mathbf{X}}^{\mathbf{i}',T}$ so that their laws are as described in Section 2.3. Let $\tilde{Y}_{i_l,s}$ be the analogue of $Y_{i_l,s}$ associated with the graphical representation for i_l. Now we construct $\mathbf{X}^{\mathbf{i},T}$ as follows. Initially we have $k+1$ families with family heads $i_1,\dots,i_k,\mathbf{i}'$. Members of each family follow the graphical representation corresponding to the family head, but whenever two members with family heads $j\prec j'$ come within distance 3, we merge the two families and declare j to be the new family head. By the above construction, $Y_{i_l,s}=\tilde{Y}_{i_l,s}$ if the members in i_l-family do not come within distance 3 of some member from other families. Now if we let

$$\mathfrak{E}_j:=\{i_j\text{-family comes within distance 3 of some other family before time } s\},\ \mathfrak{E}:=\cap_{j=1}^k\mathfrak{E}_j,$$

then it is easy to see that there is a random variable $\mathfrak{R}_j$, which is independent of $\tilde{Y}_{i_j,s}$, so that $Y_{i_1,s}\cdots Y_{i_{2k},s}\mathbf{1}_{\mathfrak{E}_j^c}=\tilde{Y}_{i_j,s}\mathfrak{R}_j\mathbf{1}_{\mathfrak{E}_j^c}$. $\tilde{Y}_{i_j,s}$ is also independent of the events $\mathfrak{E}_l, 1\le l\le k$, and

$E_{G_n,\lambda_n}\tilde{Y}_{i_j,s} = 0$. Hence

$$E_{G_n,\lambda_n}[Y_{i_1,s}\cdots Y_{i_{2k},s}\mathbf{1}_{\mathfrak{E}^c}] = \sum_{j=1}^{k} E_{G_n,\lambda_n}\left[\tilde{Y}_{i_j,s}\cdot \mathfrak{R}_j\ \mathbf{1}_{\mathfrak{E}_j^c\setminus\cup_{l=1}^{j-1}\mathfrak{E}_l^c}\right] = 0.$$

The above estimate together with the fact $|\tilde{Y}_{i_j,s}| \le 1$ implies

$$|E_{G_n,\lambda_n}[Y_{i_1,s}\cdots Y_{i_{2k},s}]| \le P_{G_n,\lambda_n}(\mathfrak{E}) \text{ for all } (i_1,\dots,i_{2k})\in W. \tag{4.10}$$ productbd

Combining (4.8), (4.9) and (4.10) and using Proposition 4.1 to estimate $P_{G_n,\lambda_n}(\mathfrak{E})$,

$$\sup_{0\le s\le T} E_{G_n,\lambda_n}U_{n,s}^{2k} \le k(2k)!n^{3k/2+1/2\varpi_0} + n^{2k}c_{4.1}(n/\lambda_n)^{-\sqrt{k}\rho_{4.1}/\varpi_0} \le 2n^{2k}c_{4.1}(n/\lambda_n)^{-\sqrt{k}\rho_{4.1}/\varpi_0}.$$

The above estimate along with Markov inequality

$$P_{G_n,\lambda_n}\left(\sup_{0\le s\le T}\left||\boldsymbol{\xi}_s^{\lambda_n}| - E_{G_n,\lambda_n}|\boldsymbol{\xi}_s^{\lambda_n}|\right| > \delta n\right) \le (\delta n)^{-2k}E_{G_n,\lambda_n}\left(\sup_{0\le s\le T}U_{n,s}^{2k}\right)$$

gives the desired result for $C_{4.2} = 2c_{4.1}\delta^{-2k}$ and $\rho_{4.2} = \rho_{4.1}/\varpi_0$. ■

main **Theorem 4.1.** *Suppose $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$. Let $\{\xi_t : t\ge 0\}$ be the latent voter model with parameter λ_n on the random r-regular graph having distribution $\tilde{\mathbb{P}}$ such that ξ_0 has product measure on $\{0,1\}^{[n]}$ with $P(\xi_0(v) = 0) = p \in (0,1)$. There is a good set of graphs $\mathcal{G}_n$ with $\tilde{\mathbb{P}}(G_n\in\mathcal{G}_n)\to 1$ so that if n is large and $G_n\in\mathcal{G}_n$, then for any $\delta\in(0,1/2), b<\infty$ and for some constants $T = T(\delta,p), T'' = T''(\delta), C_{4.1}(b,\delta) > 0$,*

$$P_{G_n,\lambda_n}\left(\frac{1}{n}|\boldsymbol{\xi}_s| \notin [1/2-\delta, 1/2+\delta] \text{ for some } s\in[\lambda_n T, \lambda_n T + n(n/\lambda_n)^b T'']\right) \le C_{4.1}(n/\lambda_n)^{-b}.$$

Proof. Let $\boldsymbol{\xi}_t^{\lambda_n} = \boldsymbol{\xi}_{\lambda_n t}$, $\mathcal{G}_n$ be as in (2.4) and $u(\cdot)$ be as in (3.2) with $u(0) = p$ and choose T large enough so that $|u(T) - 1/2| \le \delta/16$. Then we invoke Theorem 3.1 to have $|(1/n)E_{G_n,\lambda_n}\boldsymbol{\xi}_T^{\lambda_n} - (1/2)| \le \delta/8$ for large enough n. Combining the above estimate with Lemma 4.2

$$P_{G_n,\lambda_n}\left(\left|\frac{1}{n}\left|\boldsymbol{\xi}_T^{\lambda_n}\right| - 1/2\right| > \delta/4\right) \le C_{4.2}(n/\lambda_n)^{-\sqrt{k}\rho_{4.2}(\eta,T)} \tag{4.11}$$ consensusbd1

for any $k\in\mathbb{N}$ and large enough n. Now let $\Theta_0 = T$ and for $i\ge 1$,

$$\begin{aligned}\Pi_i &:= \inf\left\{t > \Theta_{i-1} : \frac{1}{n}\left|\boldsymbol{\xi}_t^{\lambda_n}\right| \in \{1/2-\delta/2, 1/2+\delta/2\}\right\},\\ \Theta_i &:= \inf\left\{t > \Pi_i : \frac{1}{n}\left|\boldsymbol{\xi}_t^{\lambda_n}\right| \in \{1/2-\delta/4, 1/2+\delta/4, 1/2-\delta, 1/2+\delta\}\right\}.\end{aligned}$$

Also let $\tilde{u}(\cdot)$ be a solution of the ODE in (3.1) with initial value $\tilde{u}(0)\in\{1/2-\delta/2, 1/2+\delta/2\}$, and T', T'' be such that $|\tilde{u}(T') - 1/2| = \delta/16$ and $|\tilde{u}(T'') - 1/2| = 15\delta/32$. A little algebra shows that $\Pi_i + T'' \le \Theta_i$ and $(1/n)|\boldsymbol{\xi}_{\Theta_i}^{\lambda_n}| \in \{1/2-\delta/4, 1/2+\delta/4\}$ if $||\boldsymbol{\xi}_{\Pi_i+s}^{\lambda_n}|/n - \tilde{u}(s)| \le 3\delta/16$ for all $s\in[0,T']$. Now using Markov property of P_{G_n,λ_n} and Theorem 3.1, $\sup_{0\le s\le T'}|(1/n)E_{G_n,\lambda_n}\boldsymbol{\xi}_{\Pi_i+s}^{\lambda_n} - \tilde{u}(s)| \le \delta/16$ for large enough n, and applying Lemma 4.2

$$\begin{aligned}&P_{G_n,\lambda_n}\left(\Theta_i \le \Pi_i + T'' \text{ or } (1/n)|\boldsymbol{\xi}_{\Theta_i}^{\lambda_n}| \notin \{1/2-\delta/4, 1/2+\delta/4\}\right)\\ \le &P_{G_n,\lambda_n}\left(\left|\frac{1}{n}|\boldsymbol{\xi}_{\Pi_i+s}^{\lambda_n}| - \tilde{u}(s)\right| > 3\delta/16 \text{ for some } s\in[0,T']\right) \le C_{4.2}(n/\lambda_n)^{-\sqrt{k}\rho_{4.2}(\eta,T')}\end{aligned} \tag{4.12}$$ consensusbd2

for any $k\in\mathbb{N}$.

Noting that if $|\boldsymbol{\xi}_s^{\lambda_n}|/n \notin \{1/2-\delta, 1/2+\delta\}$ for some $s \in [T, T+T''(n/\ ambda_n)^b]$, then either the event of (4.11) occurs, or the event in the left hand side of (4.12) occurs for some $i \le (n/\lambda_n)^b$. So if we choose k and $C_{4.1}$ so that the estimates in (4.11) and (4.12) are at most $C_{4.1}n^{-b}/2$ and $C_{4.1}n^{-2b}/2$ respectively, the proof is complete by taking union bound of the above events. ∎

5. Random walk estimates

In this section, we study some hitting times involving two random walks on $G_{v,\lceil (1/5)\log_{r-1} n\rceil}$. We begin with some simple random walks on $\mathbb{Z}_+$, which will be useful in what follows.

Deltalemma

Lemma 5.1. *Suppose $\Delta_1, \Delta_2, \dots$ are i.i.d. with $P(\Delta_1 = -1) = 1/r = 1 - P(\Delta_1 = 1)$. There is a function $\alpha_1(r,\gamma) \ge 1$, which is nonincreasing in the second argument, such that if $k \ge \alpha_1$, then*

$$P\left(\sum_{i=11}^{km} \Delta_i < m/\gamma\right) \le (r-1)^{-m/\gamma}.$$

Proof. (1) If we let $I_\Delta(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \log(Ee^{\theta\Delta_1})\}$ be the large deviation rate function, then $I_\Delta((r-2)/r) = 0, I_\Delta(0) > 0$ and $I_\Delta(\cdot)$ is a decreasing continuous function on $[0,(r-2)/r]$, as $E\Delta_1 = (r-2)/r$. Let

$$\alpha_1(r,\gamma) := \inf\{k \ge 1 : \log(r-1)/\gamma k \le I_\Delta(1/\gamma k)\} \text{ so that } \frac{r}{\gamma(r-2)} < \alpha_1 < \infty.$$

For any $k \ge \alpha_1$ we use the standard large deviation argument to get the required estimate. As $I_\Delta(1/\gamma k)$ and $1/\gamma k$ are increasing and decreasing functions of γ, α_1 is decreasing in γ.
(2) It follows by using the optional stopping theorem for the martingale $(r-1)^{-\sum_{i=1}^m \Delta_i}$ and the stopping time $T^\Delta_{-l} \wedge T^\Delta_{l'}$. ∎

We will now analyze some hitting times of a simple random walk on a certain finite graph. Recall the definitions in (2.2) and (2.3), and assume that $v \in L_1(G_n)$. Also let $\{\bar{\mathbf{S}}_m = (\bar S_m^1, \bar S_m^2) : m \ge 0\}$ be the standard discrete time coalescing random walk system of two particles, where at each step one of the particles is chosen at random and is allowed to jump to a uniform neighbor until they coalesce, and after coalescence they say put with probability 1/2 and jump to an uniform neighbor otherwise. Note that for $i = 1, 2$, $\{\bar S_m^i : m \ge 0\}$ is a lazy simple random walk. Here we will study the associated hitting times

$$\bar T^i := \inf\{m \ge 0 : d(v, \bar S_m^i) = \lceil (1/5)\log_{r-1} n\rceil\}, \bar T_\varpi := \inf\{m \ge 0 : d(\bar S_m^1, \bar S_m^2) = \lceil \varpi \rceil\} \text{ for } \varpi \ge 0. \tag{5.1}$$

Tbar

For notational convenience we say that

$$v \text{ is a midpoint of } (u_1,u_2) \text{ if (i) } d(u_1,v) = \lfloor d(u_1,u_2)/2\rfloor \text{ and (ii) } d(u_2,v) = \lceil d(u_1,u_2)/2\rceil. \tag{5.2}$$

midpoint

Tbar est

Proposition 5.2. *Let $\{\bar{\mathbf{S}}_m : m \ge 0\}$ be the random walk (as described above) such that $\bar{\mathbf{S}}_0$ $v \in L_1(G_n)$ is a midpoint of $\bar{\mathbf{S}}_0$ in the sense of (5.2).*

(1) For any $\gamma, a, b > 0$ there is a constant $\alpha = \alpha(r,\gamma,a,b) > 0$ such that if K and $\{\upsilon_n\}$ satisfy (i) $K \ge \alpha$ (ii) $\upsilon_n \to \infty$ as $n \to \infty$ (iii) $d(\bar S_0^1, \bar S_0^2) = \lceil \vartheta\upsilon_n\rceil$ for some $\vartheta \ge b$ and $[\vartheta/2 + K]\upsilon_n \le (1/5)\log_{r-1} n$, then

$$P(\bar T_{(\vartheta+a)\upsilon_n} < \bar T_{(\vartheta-b)\upsilon_n} \wedge \lceil K\upsilon_n\rceil) \ge 1 - 6(r-1)^{-\upsilon_n/\gamma} - (r-1)^{1-b\upsilon_n} - (r-1)^{-(a\wedge b)\upsilon_n},$$

(2) *For any* $\gamma > 0$ *there is a constant* $\tilde{\alpha} = \tilde{\alpha}(r,\gamma) \geq 2$ *such that if* K *and* $\{\upsilon_n\}$ *satisfy (i)* $K \geq \tilde{\alpha}$ *(ii)* $\upsilon_n \to \infty$ *as* $n \to \infty$ *and (iii)* $(3K + 1/\gamma)\upsilon_n \leq (2/5)\log_{r-1} n$*, and (iv)* $d(\bar{S}_0^1, \bar{S}_0^2) \leq (K + 1/\gamma)\upsilon_n$*, then*

$$P\left(\left\{d\left(\bar{S}^1_{\lceil K\upsilon_n\rceil}, \bar{S}^2_{\lceil K\upsilon_n\rceil}\right) < \upsilon_n/\gamma\right\} \cap \{\bar{T}_0 > \lceil K\upsilon_n\rceil\}\right) \leq 7(r-1)^{-\upsilon_n/\gamma}.$$

Proof of Proposition 5.2 for $v \in L_1(G_n) \setminus L_0(G_n)$*.* (1) $v \in L_0(G_n)$ implies that $G_{v,\lceil(1/5)\log_{r-1} n\rceil}$ is a finite r-tree. Let

$$\begin{aligned} \alpha(r,\gamma,a,b) &:= \alpha_1(r, 1/a), \text{ where } \alpha_1 \text{ is defined in Lemma 5.1, and} \\ \bar{T} &:= \bar{T}_{(\vartheta-b)\upsilon_n} \wedge \bar{T}_{(\vartheta+b)\upsilon_n} \wedge \lceil K\upsilon_n\rceil. \end{aligned}$$

We begin by estimating the probability $P(\bar{T} = \lceil K\upsilon_n\rceil)$. Observe that until time $\bar{T}_0 \wedge \bar{T}^1 \wedge \bar{T}^2$, $\{d(\bar{S}^1_m, \bar{S}^2_m) : m \geq 0\}$ is a random walk on $\mathbb{Z}_+$ with i.i.d. increments having common distribution same as that of Δ_1 of Lemma 5.1. Also $\bar{T}^1 \wedge \bar{T}^2 \geq K\upsilon_n$, because for any $m \leq K\upsilon_n$,

$$d(v, \bar{S}^i_m) \leq d(v, \bar{S}^i_0) + K\upsilon_n \leq \vartheta\upsilon_n/2 + K\upsilon_n \leq (1/5)\log_{r-1} n \text{ by (iii) and (iv).}$$

So $\bar{T} = \lceil K\upsilon_n\rceil$ implies that the increment of the above random walk after $\lceil K\upsilon_n\rceil$ many steps is at most $a\upsilon_n$. Since $K \geq \alpha_1(r, 1/a)$ by (i), Lemma 5.1 implies

$$P(\bar{T} = \lceil K\upsilon_n) \leq P\left(\sum_{i=1}^{\lceil K\upsilon_n\rceil} \Delta_i \leq a\upsilon_n\right) \leq (r-1)^{-a\upsilon_n}. \tag{5.3}$$ Hit bd4

Next we need to bound $P(\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n})$. Applying the optional stopping theorem for the stopping time $\bar{T}$ and the martingale $\bar{M}_{m\wedge\lceil K\upsilon_n\rceil\wedge\bar{T}_0}$, where $\bar{M}_m := (r-1)^{-d(\bar{S}^1_m, \bar{S}^2_m)}$,

$$\begin{aligned} (r-1)^{-\vartheta\upsilon_n} \; = \; E\bar{M}_0 \; = \; E\bar{M}_{\bar{T}} \quad &\geq \quad (r-1)^{-(\vartheta-b)\upsilon_n} P(\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n}) \\ &\quad + (r-1)^{-(\vartheta+a)\upsilon_n}[1 - P(\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n})]. \end{aligned}$$

Rearranging the above inequality,

$$P(\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n}) \leq \frac{(r-1)^{-\vartheta\upsilon_n} - (r-1)^{-(\vartheta+a)\upsilon_n}}{(r-1)^{-(\vartheta-b)\upsilon_n} - (r-1)^{-(\vartheta+a)\upsilon_n}} \leq (r-1)^{-b\upsilon_n}. \tag{5.4}$$ Hit bd6

Combining (5.3) and (5.4) avd noting that $\bar{T}_{(\vartheta+a)\upsilon_n} \geq \bar{T}_{(\vartheta-b)\upsilon_n} \wedge \lceil K\upsilon_n\rceil$ implies either $\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n}$ or $\bar{T} = \lceil K\upsilon_n\rceil$, the proof of (1) is complete.
(2) Note that $d(v, \bar{S}^1_0), d(v, \bar{S}^2_0) \leq d(\bar{S}^1_0, \bar{S}^2_0)/2$ which is in turn $\leq (1/5)\log_{r-1} n - K\upsilon_n$ by our hypothesis. This ensures that $\bar{T}^1 \wedge \bar{T}^2 \geq K\upsilon_n$. So using similar argument which leads to (5.3) and letting $\tilde{\alpha}(r,\gamma) := \alpha_1(r,\gamma)$, where α_1 is defined in Lemma 5.1, we have

$$P\left(\left\{d\left(\bar{S}^1_{\lceil K\upsilon_n\rceil}, \bar{S}^2_{\lceil K\upsilon_n\rceil}\right) < \frac{\upsilon_n}{\gamma}\right\} \cap \{\bar{T}_0 > \lceil K\upsilon_n\rceil\}\right) \leq P\left(\sum_{i=1}^{\lceil K\upsilon_n\rceil} \Delta_i < \frac{\upsilon_n}{\gamma}\right) \leq (r-1)^{-\upsilon_n/\gamma}.$$

■

For $v \in L_1(G_n) \setminus L_0(G_n)$ we need some additional machinery to prove Proposition 5.2. Let

$\mathbb{L}_v$ be the subset of $D(v, \lceil(1/5)\log_{r-1} n\rceil)$ consisting of the vertices in the loop, and (5.5) bLdef
$\mathfrak{p}_v(u) \in \mathbb{L}_v$ be the vertex in the loop nearest to $u \in D(v, \lceil(1/5)\log_{r-1} n\rceil)$.

We need to study $\{d(\bar{S}^i_m, \mathfrak{p}(\bar{S}^i_m)) : m \geq 0\}$. Note that until $\bar{S}^i_m$ hits the boundary of the graph $G_{v,\lceil(1/5)\log_{r-1} n\rceil}$, $\{d(\bar{S}^i_m, \mathfrak{p}(\bar{S}^i_m)) : m \geq 0\}$ is a lazy asymmetric random walk on $\mathbb{Z}_+$ with a little different behavior when it hits 0. Let $\{\tilde S_m : m \ge 0\}$ be a discrete time asymmetric random walk on $\mathbb{Z}_+$ having step distribution same as that of $d(\bar S^i_m, \mathfrak{p}(\bar S^i_m))$, *i.e.*,

$$P(\tilde S_{m+1} = 0|\tilde S_m = 0) = 1/2 + 2/2r = 1 - P(\tilde S_{m+1} = 1|\tilde S_m = 0)$$

$$\text{and for } k \ge 1, \quad P(\tilde S_{m+1} = k'|\tilde S_m = k) = \begin{cases} 1/2 & \text{if } k' = k \\ 1/2r & \text{if } k' = k-1 \\ (r-1)/2r & \text{if } k' = k+1 \end{cases} \tag{5.6}$$

asymRW

The following facts about this random walk will be required in what follows.

Hitting time

Lemma 5.3. *Let $\{\tilde S_m : m \ge 0\}$ be a discrete time simple random walk with transitions as in (5.6), and $\Upsilon = \inf\{m \ge 0 : \tilde S_m = 0\}$.*

(1) Then $P(\Upsilon < \infty|\tilde S_0 = 1) = 1/(r-1)$.

(2) Moreover, if $\tilde\Upsilon_1, \tilde\Upsilon_2, \dots$ are iid with common distribution given by $P(\tilde\Upsilon_1 = \cdot) = P(\Upsilon = \cdot|\Upsilon < \infty, \tilde S_0 = 1)$, then for any $\gamma > 0$ there is a constant $\alpha_2(r,\gamma) > 2r/\gamma(r-2)$ such that

$$P(\tilde\Upsilon_1 + \cdots + \tilde\Upsilon_{k/\gamma} > \alpha_2(r,\gamma)k) \le (r-1)^{-k/\gamma}.$$

Proof. Let $\varphi_\Upsilon(\theta) = E[\exp(\theta\Upsilon \mathbf{1}_{\{\Upsilon<\infty\}}|\tilde S_0 = 1)]$. Conditioning on $\tilde S_1$ and solving the resulting quadratic equation and ignoring the impossible root,

$$\varphi_\Upsilon(\theta) = \frac{1 - e^\theta/2 - \sqrt{(1-e^\theta/2)^2 - (r-1)e^{2\theta}/r^2}}{(r-1)e^\theta/r} \text{ for } \theta < \log(r/[r/2 + \sqrt{r-1}]).$$

Clearly $P(\Upsilon < \infty|\tilde S_0 = 1) = \lim_{\theta\uparrow 0}\varphi_\Upsilon(\theta) = 1/(r-1)$. Hence if $\varphi_{\tilde\Upsilon}(\theta) := Ee^{\tilde\Upsilon_1}$, then $\varphi_{\tilde\Upsilon}(\theta) = (r-1)\varphi_\Upsilon(\theta)$. Also $E(\tilde\Upsilon_1) = \varphi'_{\tilde\Upsilon}(0) = 2r/(r-2)$. So if we let

$$\alpha_2(r,\gamma) := \frac{1}{\gamma}\inf\{x : I_{\tilde\Upsilon}(x) > \log(r-1)\}, \text{ where } I_{\tilde\Upsilon}(x) := \sup_{\theta\in\mathbb{R}}\left\{\theta x - \log\varphi_{\tilde\Upsilon}(\theta)\right\}$$

is the large deviation rate function for $\tilde\Upsilon_1$, then $\alpha_2(r,\gamma) > 2r/\gamma(r-2)$ as $I_{\tilde\Upsilon}(2r/(r-2)) = 0$. For this choice of α_2, (2) follows by using standard large deviation argument. ∎

Now we use Lemma 5.3 to show that the behavior of the random walk $\tilde S_m$ at 0 does not slow it down too much.

ance traveled

Lemma 5.4. *Let $\{\tilde S_m : m \ge 0\}$ be a discrete time simple random walk with transitions as in (5.6). For any $\gamma > 0$ there is a constant $\alpha_3(r,\gamma) > 4r/\gamma(r-2)$ such that $P(\tilde S_m = 0$ for some $m > \alpha_3(r,\gamma)L) \le 3(r-1)^{-L/\gamma}$.*

Proof. Suppose $\tilde S_0 = k$. Let $\Upsilon_i, 1 \le i \le k$, be the time required for the random walk to come to $k-i$ from $k-i+1$. Also let $\Psi_j(\text{or}\Upsilon_{k+j}), j \ge 1$, be the time it takes to return to 1 (or 0) after its j^{th} (or $(j+1)^{th}$) visit to 1 (or 1). It is easy to check that (i) Ψ_is are iid and the common distribution is geometric with success probability $(r-2)/2r$, (ii) Υ_is are i.i.d. and the common distribution is same as that of $\Upsilon|\{\tilde S_0 = 1\}$ of Lemma 5.3, and (iii) Ψ_is and Υ_is are independent. Now let

$$\varsigma := \min\{i \ge 1 : \Upsilon_i = \infty\} \text{ so that } \tilde S_m \ge 1 \text{ for all } m \ge \sum_{i=1}^{\varsigma-1}\Upsilon_i + \sum_{i=1}^{\varsigma}\Psi_i.$$

To estimate the above sum first note that

$$P(\varsigma > L/\gamma) \le (r-1)^{-L/\gamma} \tag{5.7}$$

varsigmabd

by (1) of Lemma ??. On the other hand, if $\varsigma \le L/\gamma$, then (2) of Lemma 5.3 with $\alpha_2 = \alpha_2(r,\gamma)$ and the independence of Υ_is suggest that

$$\begin{aligned} P\left(\sum_{i=1}^{\varsigma-1} \Upsilon_i > \alpha_2 L, \varsigma \le L/\gamma\right) &= \sum_{i=2}^{L/\gamma} P\left(\sum_{j=1}^{i-1} \Upsilon_j > \alpha_2 L \middle| \Upsilon_j < \infty \forall 1j \le i-1\right) P(\varsigma = i) \\ &\le P\left(\sum_{j=2}^{L/\gamma} \Upsilon_j > \alpha_2 L \middle| \Upsilon_j < \infty \forall j \le L/\gamma\right) \le (r-1)^{-L/\gamma} \end{aligned} \tag{5.8}$$

In addition, if we let $\varphi_\Psi(\theta) := Ee^{\theta\Psi_1} < \infty$ for $\theta < \log(2r/(r+2))$ and

$$\alpha_4(r,\gamma) := \frac{1}{\gamma}\inf\{x : I_\Psi(x) > \log(r-1)\}, \text{ where } I_\Psi(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \log\varphi_\Psi(\theta)\}$$

is the large deviation rate function for Ψ_is, then $\alpha_4(r,\gamma) > 2r/\gamma(r-2)$, as $I_\Psi(2r/(r-2)) = 0$. Once again using standard large deviation argument,

$$P\left(\sum_{i=1}^{\varsigma} \Psi_i > \alpha_4 L, \varsigma \le L/\gamma\right) \le P\left(\sum_{i=1}^{L/\gamma} \Psi_i > \alpha_4 L\right) \le (r-1)^{-L/\gamma}. \tag{5.9}$$ Psildp

Combining (5.7), (5.8) and (5.9), and taking $\alpha_3 := \alpha_2 + \alpha_4$, the desired result follows. ■

Proof of Proposition 5.2 for $v \in L_1(G_n) \setminus L_0(G_n)$. (1) Let $\alpha(r,\gamma,a,b) := 2\alpha_1(r, 2/(a+b)) + \alpha_3(r,\gamma)$, where α_1 and α_3 are as in Lemma 5.1 and 5.4 respectively, and $\bar{T} := \bar{T}_{(\vartheta-b)\upsilon_n} \wedge \bar{T}_{(\vartheta+b)\upsilon_n} \wedge \lceil K\upsilon_n\rceil$. Recalling the observation made just before (5.6) and noting that $\bar{T}^1 \wedge \bar{T}^2 \ge K\upsilon_n$ (as argued in the display before (5.3)),

$$\text{if } H_n := \cup_{i=1}^2\{\bar{S}_m^i \in \mathbb{L}_v \text{ for some } m \ge \alpha_3(r,\gamma)\upsilon_n\}, \text{ then } P(H_n) \le 6(r-1)^{-\upsilon_n/\gamma}. \tag{5.10}$$ Kbd

by Lemma 5.4. Next we estimate $P(\{\bar{T} = K\upsilon_n\} \cap H_n^c)$. On the event H_n^c if $\bar{T}_0 > \alpha_3\upsilon_n$, then $\{d(\bar{S}_m^1, \bar{S}_m^2) : \alpha_3\upsilon_n \le m \le \bar{T}_0 \wedge \lceil K\omega_n\rceil\}$ is a random walk on $\mathbb{Z}$ with i.i.d. increments having common distribution same as that of Δ_1 of Lemma 5.1, as $\bar{T}^1 \wedge \bar{T}^2 \ge K\upsilon_n$. So $\bar{T} = \lceil K\upsilon_n\rceil$ implies that the increment of the above random walk after $(K-\alpha_3)\upsilon_n$ many steps is at most $(a+b)\upsilon_n$. Since $(K-\alpha_3)/2 = \alpha_1(r, 2/(a+b))$ by our choice, we can use Lemma 5.1 with $m = 2\upsilon_n$ to have

$$P(\{\bar{T} = K\upsilon_n\} \cap H_n^c) \le P\left(\sum_{i=1}^{(K-\alpha_3(r,\gamma))\upsilon_n} \Delta_i < (a+b)\upsilon_n\right) \le (r-1)^{-(a+b)\upsilon_n/2}. \tag{5.11}$$ Tbar est3

Finally we estimate $P(\{\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n}\} \cap H_n^c)$. We say that $\bar{\mathbf{S}}_k$ is (i) 'bad' if either $\bar{S}_k^1 \in \mathbb{L}_v, \bar{S}_k^2 \notin \mathbb{L}_v$ and $d(\bar{S}_k^1, \mathfrak{p}(\bar{S}_k^2)) = \lfloor|\mathbb{L}_v|/2\rfloor$ or $\bar{S}_k^1 \notin \mathbb{L}_v, \bar{S}_k^2 \in \mathbb{L}_v$ and $d(\mathfrak{p}(\bar{S}_k^1), \bar{S}_k^2) = \lfloor|\mathbb{L}_v|/2\rfloor$ (ii)'very bad' if $\bar{S}_k^1, \bar{S}_k^2 \in \mathbb{L}_v$ and $d(\bar{S}_k^1, \bar{S}_k^2) = \lfloor|\mathbb{L}_v|/2\rfloor$. It is easy to see that if Δ_1 is as in Lemma 5.1 and

$$\tilde{\Delta} = \begin{cases} -1, 0 \text{ and } +1 \text{ with probabilities } 1/r, 1/r \text{ and } (r-2)/r \text{ respectively} & \text{if } |\mathbb{L}_v| \text{ is odd} \\ -1 \text{ and } +1 \text{ with probabilities } 2/r \text{ and } (r-2)/r \text{ respectively} & \text{if } |\mathbb{L}_v| \text{ is even} \end{cases}$$

then the k^{th} increment

$$d(\bar{S}_{k+1}^1, \bar{S}_{k+1}^2) - d(\bar{S}_k^1, \bar{S}_k^2) \stackrel{d}{=} \begin{cases} \tilde{\Delta} & \text{if } \bar{\mathbf{S}}_k \text{ is very bad} \\ \tilde{\Delta} \text{ or } \Delta_1 \text{ with probability } 1/2 & \text{if } \bar{\mathbf{S}}_k \text{ is bad} \\ \Delta_1 & \text{otherwise.} \end{cases}.$$

We call the k^{th} increment to be 'good' (or 'bad') if it has law same as that of Δ_1 (or $\tilde{\Delta}$). Also note that (i) every bad increment of -1 is followed by a good increment, (ii) on the event H_n^c, the number of bad increments is at most $\alpha_3 \upsilon_n$ and (iii) if $\bar{\mathbf{S}}_k$ is bad, then $d(\bar{S}_k^1, \bar{S}_k^2) \geq \lfloor |\mathbb{L}_v|/2 \rfloor$. Combining these observations, coupling $\bar{\mathbf{S}}_k$ with another random walk whose increment has law same as that of Δ_1 and using optional stopping theorem,

$$\begin{aligned} P(\{\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n}\} \cap H_n^c) &\leq P\left(d(\bar{S}_0^1, \bar{S}_0^2) + \sum_{i=1}^m \Delta_i \text{ hits } (\vartheta - b)\upsilon_n + 1 \text{ before } (\vartheta + a + \alpha_3)\upsilon_n\right) \\ &\leq \frac{(r-1)^{-\vartheta\upsilon_n} - (r-1)^{-(\vartheta+a+\alpha_3)\upsilon_n}}{(r-1)^{-(\vartheta-b)\upsilon_n - 1} - (r-1)^{-(\vartheta+a+\alpha_3)\upsilon_n}} \leq (r-1)^{-b\upsilon_n+1}.. \qquad (5.12)\end{aligned}$$

Combining (5.10), (5.11) and (5.12) and noting that $\bar{T}_{(\vartheta+a)\upsilon_n} \geq \bar{T}_{(\vartheta-b)\upsilon_n} \wedge \lceil K\upsilon_n \rceil$ implies that either $\bar{T} = \bar{T}_{(\vartheta-b)\upsilon_n}$ or $\bar{T} = \lceil K\upsilon_n \rceil$, we get the result.
(2) Using similar argument which leads to (5.11) and letting $\tilde{\alpha}(r,\gamma) := 2\alpha_1(r,\gamma) + \alpha_3(r,\gamma)$, where α_1 and α_3 are as in Lemma 5.1 and 5.4 respectively, we have

$$\begin{aligned} & P\left(\left\{d\left(\bar{S}^1_{\lceil K\upsilon_n \rceil}, \bar{S}^2_{\lceil K\upsilon_n \rceil}\right) < \upsilon_n/\gamma\right\} \cap \{\bar{T}_0 > \lceil K\upsilon_n \rceil\} \cap H_n^c\right) \\ \leq\ & P\left(\sum_{i=1}^{\lceil (K-\alpha_3)\upsilon_n \rceil} \Delta_i < \upsilon_n/\gamma\right) \leq (r-1)^{-\upsilon_n/\gamma}. \end{aligned}$$

Combining the above bound with (5.10) we get the desired result. ■

Now we use Proposition 5.2 and the coupling in Proposition 2.4 to prove the following properties of the coalescing random walk system $\hat{S}_t^{\mathbf{y}}$ (described just before (2.28)) starting from $\mathbf{y} = (y_1, y_2)$, which is the main result of this section.

dual repulsion

Proposition 5.5. *Let $\mathcal{G}_n$ be as in (2.4) and $\log n \ll \lambda_n \ll n/(\log n)^\eta$ for some $\eta > 0$. There are constants $\beta, \varpi_0(\eta) > 0$ such that*

(1) if $\omega_n = (1/\varpi)\log_{r-1}(n/\lambda_n)$ for some $\varpi \geq \varpi_0$ and $\epsilon_n := \beta\omega_n\lambda_n^{-3}(1+\lambda_n)^2$, then for any fixed $T > 0$, $G_n \in \mathcal{G}_n$ and $\mathbf{y} = (y_1, y_2)$ satisfying $d(y_1, y_2) \geq \omega_n$,

$$\begin{aligned} P_{G_n,\lambda_n}\left(d\left(\hat{S}_s^{y_1}, \hat{S}_s^{y_2}\right) < \omega_n/2 \text{ for some } s \in [0,T]\right) &\leq (r-1)^{-\rho_1(\eta)\omega_n} \text{ for some } \rho_1(\eta) > 0, \\ P_{G_n,\lambda_n}\left(d\left(\hat{S}_s^{y_1}, y_2\right) < \omega_n/2 \text{ for some } s \in [0,T]\right) &\leq (r-1)^{-\rho(\eta)\omega_n} \text{ for some } \rho(\eta) > 0, \text{ and} \end{aligned}$$

(2) if $\omega_n \leq (1/3\varpi_0)\log_{r-1}(n/\lambda_n)$ and ϵ_n is as in (1), then for any fixed $T > 0$

$$P_{G_n,\lambda_n}\left(\left\{d(\hat{S}_s^{y_1}, \hat{S}_s^{y_2}) < \omega_n \text{ for some } s \in [\epsilon_n, T]\right\} \cap \left\{\hat{S}_s^{y_1} \neq \hat{S}_s^{y_2} \forall s \in [0, \epsilon_n]\right\}\right) \to 0 \text{ uniformly in } G_n \in \mathcal{G}_n.$$

Proof. For α and $\tilde{\alpha}$ are as in Proposition 5.2 let $\beta := \tilde{\alpha}(r, 1/3)$ and

$$\begin{aligned} &\eta' = 1/4, K := \alpha(r, 8, 1/8, 1/8), \varpi_0 := \max\{5(K+1), 15(3\beta+1)/2\} && \text{if } \limsup_{n\to\infty}(\log\lambda_n/\log n) \\ &\varpi_0 = \eta/16, \eta'(\varpi) = 4\varpi/\eta, K = \max\{\alpha(r, 2/\eta', \eta' - 1/4, 3/4), \alpha(r, 2/\eta', \eta'/2, \eta'/2)\} && \text{otherwise.} \end{aligned}$$

The choices of K and ϖ_0 ensures that for $\omega_n = (1/\varpi)\log_{r-1}(n/\lambda_n), \varpi \geq \varpi_0$,

$$(i)(1/2 + K)\omega_n \leq (1/5)\log_{r-1} n \text{ for large enough } n \quad (ii)(r-1)^{(\eta'/4)\omega_n} \gg \log n, \qquad (5.13)$$

check hyp1

which will be needed in what follows.

(1) Recall the coalescing random walk system $S_t^{\lambda_n,\mathbf{y}}$ described before (2.31) and let

$$\mathbf{D} := \{(x,y) : x,y \in [n] \text{ and } d(x,y) \le 3\omega_n/4\}, T_{\mathbf{D}}^{\mathbf{y}} := \inf\{t>0 : S_t^{\lambda_n,\mathbf{y}} \in \mathbf{D}\},$$
$$\mathcal{R} := \inf\left\{t > 0 : d_{TV}\left(S_t^{\lambda_n,y_i}, U_{[n]}\right) \le 1/n^2 \text{ for each } i=1,2\right\}.$$

Note that if $\sup_{0\le s\le T} d(\hat{S}_s^{y_i}, S_s^{\lambda_n,y_i}) \le \omega_n/8$ for each $i=1,2$ and $d(\hat{S}_s^{y_1},\hat{S}_s^{y_2}) < \omega_n/2$ for some $s \in [0,T]$, then using triangle inequality $T_{\mathbf{D}}^{\mathbf{y}} \le T$. This implies

$$P_{G_n,\lambda_n}\left(d\left(\hat{S}_s^{y_1},\hat{S}_s^{y_2}\right) < \omega_n/2 \text{ for some } s\in[0,T]\right) \le P_{G_n,\lambda_n}\left(\cup_{i=1}^2 \left\{\sup_{0\le s\le T} d(\hat{S}_s^{y_i}, S_s^{\lambda_n,y_i}) \ge \omega_n/8\right\}\right)$$
$$+ P_{G_n,\lambda_n}(T_{\mathbf{D}}^{\mathbf{y}} \le \mathcal{R}) + P_{G_n,\lambda_n}(\mathcal{R} < T_{\mathbf{D}}^{\mathbf{y}} \le T) \tag{5.14}$$

break up1

By (1a) of Lemma 2.4 the first term in the right hand side of (5.14) is $\le C_1(r-1)^{-c_1\omega_n}$ for some constants $C_1, c_1 > 0$, which only depend on T. To bound the second term note that for $G_n \in \mathcal{G}_n$ if we let v be a midpoint of $\mathbf{y}$ in the sense of (5.2), then the distribution of $\{\bar{\mathbf{S}}_m^{\mathbf{Y}} : m \ge 0\}$ is same as that of the underlying discrete time jumps of $\{S_s^{\lambda_n,\mathbf{y}} : s \ge 0\}$. So using (1) of Proposition 5.2 ((i) of (5.13) ensures that the hypothesis holds) we have

$$P_{G_n,\lambda_n}\left(d\left(S_s^{\lambda_n,y_1}, S_s^{\lambda_n,y_2}\right) \text{ fails to reach } (3/4+\eta')\omega_n \text{ without hitting } 3\omega_n/4\right)$$
$$\le C_2(r-1)^{-c_2(\eta)\omega_n} \tag{5.15}$$

separation inc

for some constants $C_2, c_2 > 0$. Another application of (1) of Proposition 5.2 suggests that after reaching $((3/4+\eta')\omega_n$ whenever the distance between $S_s^{\lambda_n,y_1}$ and $S_s^{\lambda_n,y_2}$ reaches $(3/4+\eta'/2)\omega_n$, it fails to increase to $(3/4+\eta')\omega_n$ before deceasing to $3\omega_n/4$ in one attempt with probability $\le C_3(r-1)^{-\omega_n\eta'/2}$. At least $\omega_n\eta'$ steps are needed in the $S_t^{\lambda_n,\mathbf{y}}$ system between two successive occasions when the distance between the two particles equals $(3/4+\eta'/2)\omega_n$ and hits $\{(3/4+\eta')\omega_n, 3\omega_n/4\}$ in between. So

$$P_{G_n,\lambda_n}\left(d\left(S_s^{\lambda_n,y_1}, S_s^{\lambda_n,y_2}\right) \text{ hits } 3\omega_n/4 \text{ after reaching } (3/4+\eta')\omega_n \text{ before } (\omega_n\eta')(r-1)^{\omega_n\eta'/4}\right.$$
$$\left.\text{many steps are taken in } S_t^{\lambda_n,\mathbf{y}} \text{ system}\right) \le C_3(r-1)^{-\omega_n\eta'/4}. \tag{5.16}$$

Since both the particles in $S_t^{\lambda_n,\mathbf{y}}$ system are always equally likely to jump, standard large deviation argument suggests that with probability $\ge 1-(r-1)^{-c_4(\eta)\omega_n}$ each of the particles jumps at least $(\omega_n\eta'/4)(r-1)^{\omega_n\eta'/4}$ many times before $(\omega_n\eta')(r-1)^{\omega_n\eta'/4}$ many steps are taken in $S_t^{\lambda_n,\mathbf{y}}$ system.

Combining the above observations, noting that the TV distance between $U_{[n]}$ and the law of $S_t^{\lambda_n,y_i}$ reduces to $1/n^2$ before the particle jumps $O(\log n)$ times and using (ii) of (5.13),

$$P_{G_n,\lambda_n}\left(T_{\mathbf{D}}^{\mathbf{y}} \le \mathcal{R}\right) \le C_5(r-1)^{-c_5\omega_n}. \tag{5.17}$$

reach stationa

To bound the third term in the right hand side of (5.14) note that if we writing $\mathfrak{u}_n$ and $\mathfrak{u}_n^{\mathbf{D}}$ for $U_{[n]}\times U_{[n]}$ and its restriction to $\mathbf{D}$, then $P_{G_n,\lambda_n}(\mathcal{R} < T_{\mathbf{D}}^{\mathbf{y}} \le T) \le 1/n^2 + P_{G_n,\lambda_n}(T_{\mathbf{D}}^{\mathbf{u}} \le T | \mathbf{u} \sim \mathfrak{u}_n)$ by the definition of $\mathcal{R}$. So using (6.8.2) of [3], which is am implication of Proposition 23 in

Aldous and Fill (2003), to bound $P_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}} \le T | \mathbf{u} \sim \mathfrak{u}_n)$

$$P_{G_n,\lambda_n}(\mathcal{R} < T^{\mathbf{y}}_{\mathbf{D}} \le T) \le 1/n^2 + 1 - \exp\left(-\frac{T}{E_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}}|\mathbf{u} \sim \mathfrak{u}_n)}\right) + \frac{c_6}{E_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}}|\mathbf{u} \sim \mathfrak{u}_n)}$$
$$\le 1/n^2 + C_6/E_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}}|\mathbf{u} \sim \mathfrak{u}_n). \quad (5.18)$$ AldFill

Following the proof of (6.8.3) in [3] (see page 179),

$$\frac{1}{\lambda_n} \cdot \frac{1}{U_{[n]} \times U_{[n]}(\mathbf{D})} = E_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}}|\mathbf{u} \sim \mathfrak{u}^{\mathbf{D}}_n)$$
$$= o(n) + E_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}}|\mathbf{u} \sim \mathfrak{u}^{\mathbf{D}}_n, T^{\mathbf{u}}_{\mathbf{D}} \gg \log n) P_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}} \gg \log n|\mathbf{u} \sim \mathfrak{u}^{\mathbf{D}}_n)$$
$$= o(n) + E_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}}|\mathbf{u} \sim \mathfrak{u}_n) P_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}} \gg \log n|\mathbf{u} \sim \mathfrak{u}^{\mathbf{D}}_n). \quad (5.19)$$ mean hit

The extra $1/\lambda_n$ factor appears in the beginning of the above display because the particles in the $S^{\lambda_n,\cdot}_t$ system jumps at rate λ_n. Also note that $\mathfrak{u}_n(\mathbf{D}) \le r(r-1)^{\omega_n}/n(r-2)$ and $P_{G_n,\lambda_n}(T^{\mathbf{u}}_{\mathbf{D}} \gg \log n|\mathbf{u} \sim \mathfrak{u}^{\mathbf{D}}_n) = (r-2)/r + o(1)$. So combining (5.18) and (5.19) the third term in the right hand side of (5.14) is

$$\le C_7 \frac{\lambda_n (r-1)^{\omega_n}}{n} = C_7 (r-1)^{-c_7(\eta)\omega_n}$$

for some constants C_7, c_7. This completes the proof of the first assertion of (1). The proof of the other assertion is similar.
(2) Using triangle inequality it is easy to see that

$$P_{G_n,\lambda_n}\left(\left\{d(\hat{S}^{y_1}_s, \hat{S}^{y_2}_s) < \omega_n \text{ for some } s \in [\epsilon_n, T]\right\} \cap \left\{\hat{S}^{y_1}_s \ne \hat{S}^{y_2}_s \text{ for all } s \in [0, \epsilon_n]\right\}\right)$$
$$\le P_{G_n,\lambda_n}\left(\left\{d(S^{\lambda_n,y_1}_s, S^{\lambda_n,y_2}_s) < 3\omega_n/2 \text{ for some } s \in [\epsilon_n, T]\right\} \cap \left\{S^{\lambda_n,y_1}_s \ne S^{\lambda_n,y_2}_s \forall s \in [0, \epsilon_n]\right\}\right)$$
$$+ P_{G_n,\lambda_n}\left(\cup_{i=1}^2 \left\{\sup_{0 \le s \le T} d(\hat{S}^{y_1}_s, S^{\lambda_n,y_2}_s) > \omega_n/4\right\}\right)$$
$$+ P_{G_n,\lambda_n}\left(\left\{\hat{S}^{y_1}_s \ne \hat{S}^{y_2}_s \text{ for all } s \in [0, \epsilon_n]\right\} \cap \left\{S^{\lambda_n,y_1}_s = S^{\lambda_n,y_2}_s \text{ for some } s \in [0, \epsilon_n]\right\}\right) \quad (5.20)$$ break up

Using (1a) of Proposition 2.4 the second term in the right hand side of (5.20) is $o(1)$. To bound the third term note that if $d(\hat{S}^{y_1}_s, S^{\lambda_n,y_i}_s) \le 1$ for all $s \in [0, \epsilon_n]$, then the event of interest occurs only when one of the two particles $\hat{S}^{y_i}_s$ has a single voting time and the corresponding wake up dot in between a single voting time and the corresponding single wake up dot for the other particle. The probability of one such occurrence is at most $1/\lambda_n$ by (2.32), and so imitating the proof of (1b) of Proposition 2.4 the third term in the right hand side of (5.20) is at most $2(\epsilon_n + 1/\lambda_n)$.

To bound the first term in the right hand side of (5.20), let U_1 be the number of steps taken in $S^{\lambda_n,\mathbf{y}}_t$ system before time ϵ_n. Clearly U_1 has Poisson distribution with mean $2\beta\omega_n$ and

$$P_{G_n,\lambda_n}(U_1 \notin [\beta\omega_n, 3\beta\omega_n]) = o(1) \quad (5.21)$$ Ubd

by standard large deviation argument. Now note that for $G_n \in \mathcal{G}_n$ if we let v be the mid-point of $\mathbf{y}$ in the sense of (5.2), then $\{U_1 = L\omega_n\} \subset \{S^{\lambda_n,\mathbf{y}}_{\epsilon_n} \stackrel{d}{=} \bar{\mathbf{S}}_{L\omega_n}\}$ (described just before Proposition 5.2). So if $U_1 = L\omega_n$ and $d(S^{\lambda_n,y_1}_0, S^{\lambda_n,y_2}_0) > (L+3)\omega_n$ for some L, then obviously $d(S^{\lambda_n,y_1}_{\epsilon_n}, S^{\lambda_n,y_2}_{\epsilon_n}) > 3\omega_n$. Also if $U_1 = L\omega_n$ and $d(S^{\lambda_n,y_1}_0, S^{\lambda_n,y_2}_0) \le (L+3)\omega_n$ for some

$L \in [\beta, 3\beta]$, then we can apply (2) of Proposition 5.2 with $\gamma = 1/3$, because the choices of β and ϖ_0 ensure the requirements for L. These observations together with (5.21) give

$$P_{G_n,\lambda_n}\left(\left\{d\left(S^{\lambda_n,y_1}_{\epsilon_n}, S^{\lambda_n,y_2}_{\epsilon_n}\right) < 3\omega_n\right\} \cap \left\{S^{\lambda_n,y_1}_s \neq S^{\lambda_n,y_2}_s \forall s \in [0,\epsilon_n]\right\}\right) = o(1).$$

Also note that (1) of this lemma suggests

$$P_{G_n,\lambda_n}\left(\left\{d\left(S^{\lambda_n,y_1}_s, S^{\lambda_n,y_2}_s\right) < 3\omega_n/2\right\} \cap \left\{d\left(S^{\lambda_n,y_1}_{\epsilon_n}, S^{\lambda_n,y_2}_{\epsilon_n}\right) \geq 3\omega_n\right\}\right) = o(1).$$

Combining last two displays the proof of (2) is complete. ■

References

CD11 [1] S. Chatterjee and R. Durrett, *Asymptotic behavior of aldous' gossip process*, Arxiv preprint arXiv:1005.1608 (2010).

CDP11 [2] J. T. Cox, R. Durrett, and E. Perkins, *Voter Model Perturbations and Reaction Diffusion Equations*, http:.

D07 [3] R. Durrett, *Random graph dynamics*, Vol. 20, Cambridge Univ Pr, 2007.

DZ07 [4] R. Durrett and I. Zähle, *On the width of hybrid zones*, Stochastic Processes and their Applications **117** (2007), no. 12, 1751–1763.

LSB09 [5] R. Lambiotte, J. Saram
"aki, and V. D. Blondel, *Dynamics of latent voters*, Physical Review E **79** (2009), no. 4, 046107.